ANALYSIS, COMBINTORICS +

COMPUTATION

EULER NUMBER χ

$V - E + F = 2$ (ON SPHERE)

$= 2 - 2g$ (ON SURFACE OF GENUS g)

HIGHER-DIMS

$$\sum_{q=0}^{n} (-1)^q c^q = \sum_{q=0}^{n} (-1)^q h^q = \chi$$

$c^q =$ NUMBER OF q-CELLS

$h^q = \dim H^q$

$H^q =$ (CO)-HOMOLOGY GROUP

COMBINATORIAL

TOPOLOGICAL INVARIANTS

INTEGRAL FORMULA

RIEMANNIAN METRIC

dim 2 GAUSS (SCALAR) CURVATURE S

$$\chi = \frac{1}{2\pi}\int S$$

(ALSO IN HIGHER-DIMS)

DIFFERENTIAL FORMULA

Ω^q DIFFERENTIAL q-FORMS

$d : \Omega^q \to \Omega^{q+1}$ EXTERIOR DERIVATIVE

DE RHAM COMPLEX

$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^n \to 0$$

H^q = COHOMOLOGY OF THIS COMPLEX

HODGE THEORY

RIEMANNIAN METRIC

d^* = ADJOINT OF d

HODGE LAPLACIAN $\Delta = dd^* + d^*d$

$\Delta\phi = 0$ HARMONIC q-FORMS $\mathcal{H}^q$

HODGE THEOREM $\mathcal{H}^q \cong H^q$

$$\Rightarrow \chi = \sum (-1)^q \dim \mathcal{H}^q$$

LAPLACE TYPE OPERATORS

2^{ND} ORDER ELLIPTIC

IN $\mathbb{R}^N$ $\quad -\sum_1^n \frac{\partial^2}{\partial x_i^2}$ $\quad$ HODGE Δ

DIRAC TYPE OPERATORS

1^{ST} ORDER ELLIPTIC (SYSTEMS)

IN $\mathbb{R}^N$ $\quad \sum_1^n A_i \frac{\partial}{\partial x_i}$ $\quad A_i^2 = -1$

$A_i A_j = -A_j A_i$
$(i \neq j)$

SQUARE-ROOT OF
LAPLACE-TYPE

dim 1 $\quad \frac{\partial}{\partial x}$

dim 2 $\quad$ CAUCHY-RIEMANN $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

ON DIFFERENTIAL FORMS $\Omega = \bigoplus_q \Omega^q$

$\boxed{\text{SPINORS}}$

DIRAC $\quad (d + d^*)$

$(d+d^*)^2 = dd^* + d^*d$ $\quad$ HODGE Δ

$(d^2 = (d^*)^2 = 0)$

INDEX THEORY

$$d + d^* : \Omega^{ev} \to \Omega^{odd}$$

$$\underline{\text{index}} = \dim \mathcal{H}^{ev} - \dim \mathcal{H}^{odd} = \chi$$

SIMPLEST EXAMPLE OF AN INDEX THEOREM (GIVEN BY INTEGRAL FORMULA)

BUT ATYPICAL : VERY ROBUST

COMBINATORIAL VERSION PRECISE AT ALL LEVELS

IN GENERAL

1) FOR DIRAC-TYPE OPERATOR D

$\mathcal{H}^+$, $\mathcal{H}^-$ (NULL-SPACES OF D^*D, DD^*) HAVE DIMS WHICH CAN "JUMP" THOUGH difference (INDEX) IS TOPOLOGICAL INVARIANT.

2) COMBINATORIAL APPROXIMATIONS TO D "NOT PRECISE"

EXAMPLES OF (2)

a) dim 2 SURFACE CONFORMAL (COMPLEX)

$d = \partial + \bar\partial \qquad \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\right)$

COMPLEX DE RHAM

$\Omega^0 \xrightarrow{\bar\partial} \Omega^{0,1}$

$\mathcal{H}^0$ = CONSTANTS $\quad \mathcal{H}^1$ = HOLOMORPHIC DIFFLS

dim g

index = $1-g$ $\quad \left(= \frac{1}{2}\chi\right)$

BUT WHAT IS COMBINATORIAL VERSION?

b) dim 4 (ORIENTED)

H^2 HAS QUADRATIC FORM

(H_2 " INTERSECTION FORM)

+ METRIC

$H^2 = \mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$

$*\varphi = \varphi \qquad *\varphi = -\varphi$

$\boxed{\text{Signature}}$ $\sigma = \dim \mathcal{H}^2_+ - \dim \mathcal{H}^2_-$

TOPOLOGICAL INVARIANT

INTEGRAL FORMULA $= \frac{1}{3}\int p_1$

(PONTRJAGIN FORM)

σ CAN BE VIEWED AS INDEX
OF OPERATOR $d + d^*$: $\Omega^+ \to \Omega^-$
WHERE $\Omega = \Omega^+ \oplus \Omega^-$ IS
DECOMPOSITION GIVEN BY $*$

NOTE $*$ MAPS $\Omega^0 \leftrightarrow \Omega^4$
$\Omega^1 \leftrightarrow \Omega^3$

AND DECOMPOSES Ω^2 INTO SELF-DUAL
& ASD. $\Rightarrow$ index = Signature

BUT FINDING COMBINATORIAL VERSION
RUNS INTO PROBLEMS

CELLULAR $\longrightarrow$ DUAL CELLS

REFINEMENT

to DEFINE INTERSECTIONS AT CELLULAR LEVEL

INFINITE REGRESSION!

COMBINATORIAL INDEX PROBLEM

IF $D : E \to F$ ELLIPTIC (DIRAC-TYPE) DIFF. OPERATOR

WITH ITS ADJOINT D^*

HAD A PRECISE COMBINATORIAL VERSION

AT ALL LEVLS

$$D_N : E_N \to F_N$$

THEN

$$\text{index } D_N = \dim E_N - \dim F_N$$

(BECAUSE WE ARE IN FINITE DIMS)

AS $N \to \infty$ WE WANT TO RECOVER

DIFF. OPERATOR STORY

SO $\text{INDEX } D_N \to \text{INDEX } D$

IT SEEMS WE NEED TO FEED THE ANSWER INTO THE COMB. APROX

BREAKING THE SYMMETRY

GIVEN $D: E \to F$ ELLIPTIC

PERHAPS WE CAN FIND FINITE APPROXIMATIONS D_N & D'_N TO D & D^* WHICH ARE <u>NOT</u> ADJOINTS

WITH $D_N \to D$ & $D'_N \to D^*$

THEN WE NEED TO STUDY HOW

$$(D_N^* D_N) \to D^* D$$

& HOW ITS EIGENVALUES BEHAVE AS $N \to \infty$.

KEY OBSERVATION "PHYSICS"

FOCUS NOT ON 0-EIGENVALUE OF D_N BUT ON <u>LOW-LYING EIGENVALUES</u>

THOSE WHICH $\to 0$ AS $N \to \infty$

J. PACHOS (LEEDS)

COMPUTATIONAL QUESTION

HOW DO WE IDENTIFY THE "LOW-LYING" EIGENVALUES IN SOME PRECISE WAY ?

NOTE FOR DIRAC-TYPE OPERATORS ARISING IN RIEMANNIAN GEOMETRY THE KEY DATA (METRIC + CONNECTION) $\Rightarrow$ CURVATURE (OF BASE + BUNDLE)

PERHAPS WE NEED AN APPROXIMATION SO THAT ALL CURVATURES OVER CELLS ARE "SMALL" ?

(1) EXAMPLE OF "JUMPS"

dim 2 COMPLEX RIEMANN SURFACE X

HOLOMORPHIC LINE-BUNDLE L

$\dim H^0(X,L)$ CAN VARY AS COMPLEX

MODULI OF L VARY $(g \geq 1)$

BUT $\dim H^0 - \dim H^1$ TOPOLOGICAL

$= 1 + \deg(L)$

NOTE HERE VARIATION IS BOUNDED

IN HIGHER-DIMS IS UNBOUNDED

QUESTION HOW WILL THIS AFFECT

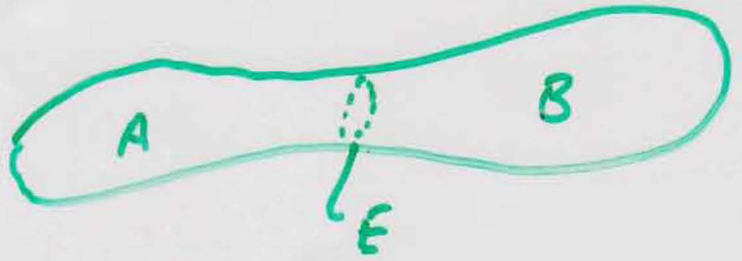
COMBINATORIAL APPROXIMATIONS ?

EIGENVALUES & CURVATURE

FOR LAPLACE-BELTRAMI OPERATOR
(= HODGE ON SCALAR FUNCTIONS)

ESTIMATES ON FIRST NON-ZERO
EIGENVALUE λ_1

CHEEGER ISOPERIMETRIC CONSTANT
M COMPACT RIEMANNIAN



M dim n

$$h(M) = \inf_{E} \left\{ \frac{S(E)}{\text{Min}\,(V(A), V(B))} \right\}$$

E dim n-1

V = n-volume
S = (n-1)-volume "area"

CHEEGER INEQUALITY

$$\lambda_1(M) \geq \frac{h^2(M)}{4}$$

"SHARP"

IF RIEMANNIAN CURVATURE

BOUNDED BELOW $\geq -(n-1)a^2 \quad (a \geq 0)$

THEN

$$\lambda_1(M) \leq 2a(n-1)h(M) + 10h^2(M)$$

BUSER INEQUALITY

STORY MORE COMPLICATED FOR OTHER

GEOMETRIC OPERATORS

PROBABLY LARGE LITERATURE ??

FOCUS ON KEY EXAMPLE OF SIGNATURE OF 4-MANIFOLD

[NOTE RESULTS OF GELFAND - MACPHERSON]

STEPS

1) SET UP COMB. APPROXIMATIONS

$$\Delta_N, \Delta'_N$$

2) DEFINE "LOW-LYING" EIGENVALUES OF THESE

3) USE CURVATURE ESTIMATES TO SHOW THESE EIGENVALUES ARE WELL-DEFINED FOR LARGE N

6) CONCLUDE

$$\text{SIGNATURE} = k_N - k'_N$$

k, k' NUMBER OF LOW-LYING EIGENVALUES OF Δ_N, Δ'_N

HOPE TO GET USEFUL FORMULA !!