Delaunay-type structures for manifolds

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GEOMETRICA
INRIA Sophia-Antipolis

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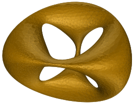
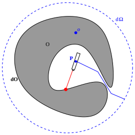
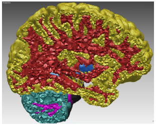
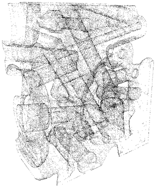
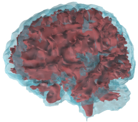
Outline

1. Delaunay triangulation and surface mesh generation
2. Stability of Delaunay triangulations
3. Approximating Riemanniann Delaunay triangulations
4. Bypassing the curse of dimensionality

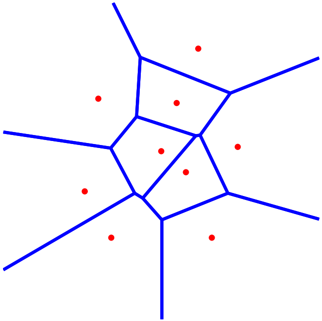
Meshing surfaces and 3D domains

- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medecine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing : meshes for FEM

Voronoi diagram and Delaunay triangulation

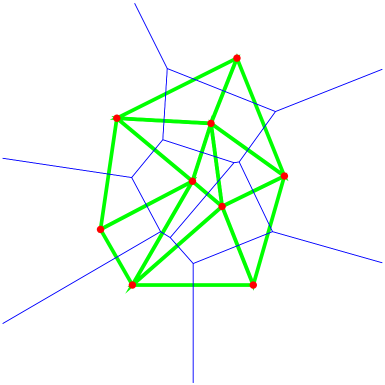
P a finite set of points in $\mathbb{R}^d$

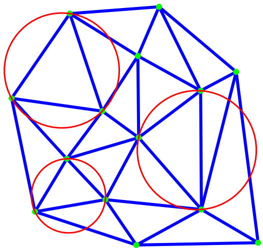


Voronoi cell $V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram $\mathrm{Vor}(\mathsf{P}) = \{ \text{ cells } V(p_i) \text{ and their faces, } p_i \in \mathsf{P} \ \}$

Voronoi diagram and Delaunay triangulation



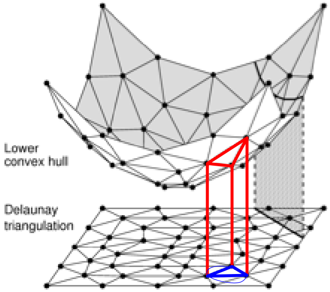


Delaunay triangulation : $\text{Del}(\text{P}) =$ nerve of $\mathrm{Vor}(\text{P})$

$\text{Del}(\text{P})$ can also be defined as the set of simplices with an empty circumscribing ball

Delaunay's fundamental result [1934]

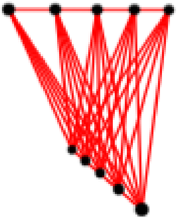
Del (P) is a simplicial complex embedded in $\mathbb{R}^d$ if P is in general position (i.e. no $d+2$ points on a same sphere)



Lifting map : $x \in \mathbb{R}^d \rightarrow (x, x^2) \in \mathbb{R}^{d+1}$

Upper Bound Theorem [Mc Mullen 1970]

faces of Del (P) = $\Theta\left(|\mathrm{P}|^{\lfloor \frac{d+1}{2} \rfloor}\right)$

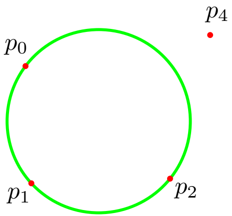


(Place P on the moment curve $\{(t, t^2, ..., t^{d+1}), t \in \mathbb{R}\}$)

Algorithms

- Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$ [Clarkson & Shor 1989] [Chazelle 1992]

- Predicate :



$$\texttt{insphere}(p_0, \ldots, p_{d+1}) = \text{sign} \begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}$$

- CGAL : $8.5s$ for 10^6 points in $\mathbb{R}^3$ (64 bit Intel Xeon 3 GHz)

The CGAL Library http://www.cgal.org

A European consortium

ETH Zurich, INRIA Sophia Antipolis, MPI Saarbrucken, Tel Aviv University...

A 15 years development project

- 600.000 lines of code
- 3 000 pages of doc
- 45 men-years
- 50 developers

Widely used for teaching, research and industry

- used in astrophysics, medecine, biology, geosciences, image processing, scientific computing etc.
- 10 000 downloads per year, 800 users
- startup GeometryFactory

Restricted Delaunay triangulation [Chew 93]

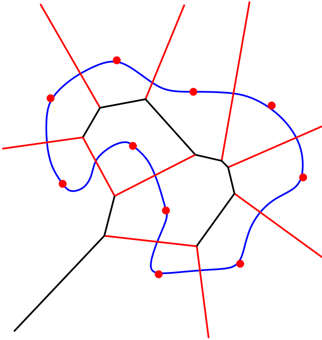
Definition

$\mathcal{M}$ a submanifold of $\mathbb{R}^d$ P $\subset \mathcal{M}$

$\mathrm{Vor}_{|\mathcal{M}}(\mathrm{P})$ = restriction of $\mathrm{Vor}(\mathrm{P})$ to $\mathcal{M}$

The Delaunay triangulation restricted to $\mathcal{M}$ $\mathrm{Del}_{|\mathcal{M}}(\mathrm{P})$ is the nerve of $\mathrm{Vor}_{|\mathcal{M}}(\mathrm{P})$

Equivalently, $\mathrm{Del}_{|\mathcal{M}}(\mathrm{P})$ is the set of simplices of Del (P) with an empty circumscribing ball centered on $\mathcal{M}$



If P is in general position, $\mathrm{Del}_{|\mathcal{M}}(\mathrm{P})$ is a simplicial complex embedded in $\mathbb{R}^d$

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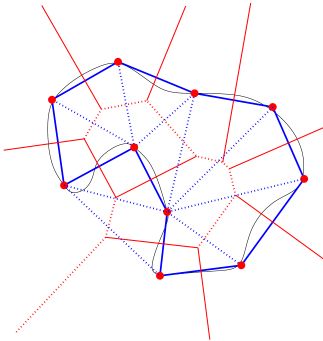

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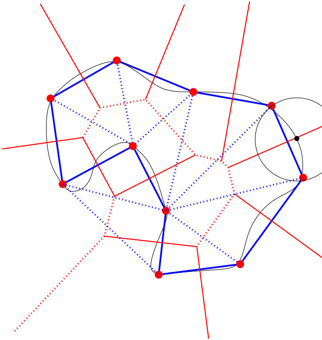

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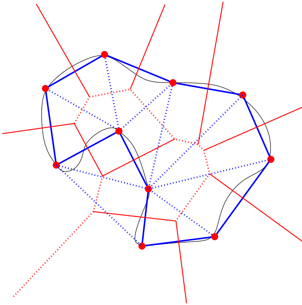
A variant of the nerve theorem

Theorem [Edelsbrunner & Shah 1997]

If $\mathcal{M}$ is compact and without boundary and if, for any face $f \in \text{Vor}_{|S}(\mathsf{P})$,

1. f intersects $\mathcal{M}$ transversally
2. $f \cap \mathcal{M} = \emptyset$ or is a topological ball of the right dimension

then $\quad \text{Del}_{|\mathcal{M}}(\mathsf{P}) \approx \mathcal{M}$



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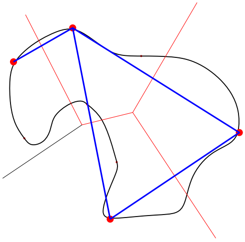
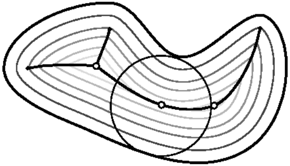


Sampling conditions [Federer 1958], [Amenta & Bern 1998]

Medial axis of $\mathcal{M}$: Med($\mathcal{M}$)

set of points with at least two closest points on $\mathcal{M}$



Reach : $\mathrm{rch}(x)$

$\forall x \in \mathcal{M}, \quad \mathrm{rch}(x) =$ infimum of the radii of the medial balls at x

ϵ-sample of $\mathcal{M}$ (ε-covering)

$\mathcal{P} \subset \mathcal{M}, \forall x \in \mathcal{M}: \qquad d(x, \mathcal{P}) \leq \epsilon \, \mathrm{rch}(x)$

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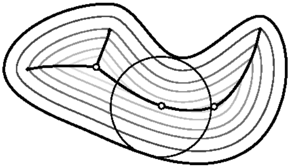

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Restricted Delaunay triangulations of smooth surfaces

[Amenta et al. 1998-]

$\mathcal{S}$ a $C^{1,1}$ surface $\subset \mathbb{R}^3$
$\mathcal{P}$ an ε-sample of $\mathcal{S}$, $\quad \varepsilon \leq 0.12$

- $\mathrm{Del}_{|S}(\mathcal{S})$ provides good estimates of
  - normals
  - areas
  - curvature [Cohen-Steiner, Morvan]
- There exists an ambient isotopy $\phi : \mathrm{Del}_{|S}(\mathcal{P}) \to \mathcal{S}$
- $\sup_x(\|\phi(x) - x\|) = O(\varepsilon^2)$

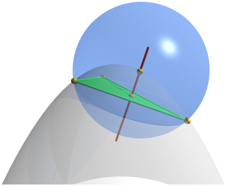
Surface mesh generation by Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\ 0 < \phi_{\min} \leq \phi(x) < \varepsilon\, \mathrm{rch}(x)$

ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$



Algorithm

INIT compute an initial (small) sample $\mathcal{P}_0 \subset S$

REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ and $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL all facets are good

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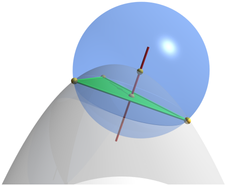

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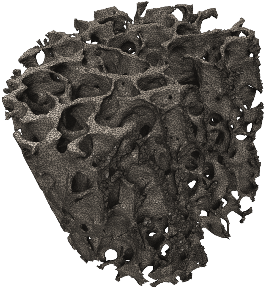
Properties of the algorithm

- The algorithm terminates and produces
  - guaranteed approximations of surfaces
  - sparse ε-samples
- The algorithm uses a black box model of surfaces
  - Contouring isosurfaces in 3D images
  - Mesh generation of volumes bounded by curved surfaces
  - Multibody mesh generation
  - Meshing 3D domains with piecewise smooth boundaries
  - Point set surfaces
- Can be downloaded from the CGAL library (www.cgal.org)

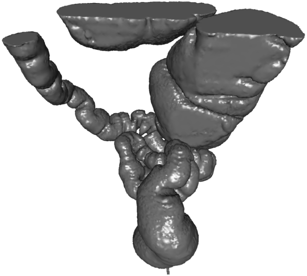
Meshing 3D domains

Input from segmented 3D medical images

[INSERM]

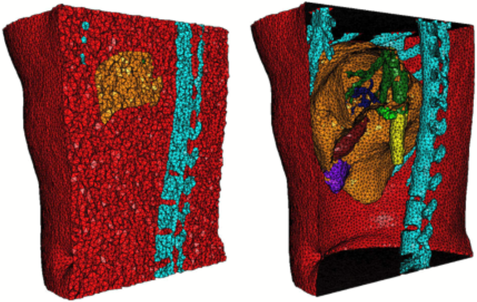


[SIEMENS]



Meshing 3D multi-domains

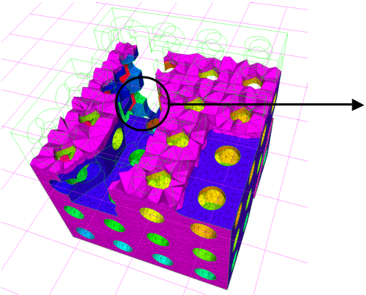
Input from segmented 3D medical images [IRCAD]

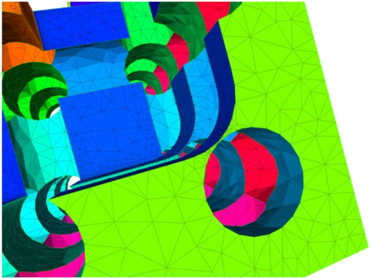


Size bound (mm)	vertices nb	facets nb	tetrahedra nb	CPU Time (s)
16	3,743	3,735	19,886	0.880
8	27,459	19,109	159,120	6.97
4	199,328	76,341	1,209,720	54.1
2	1,533,660	311,420	9,542,295	431

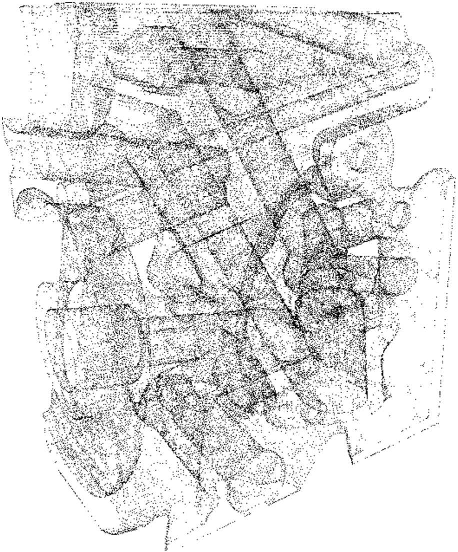
Meshing with sharp features

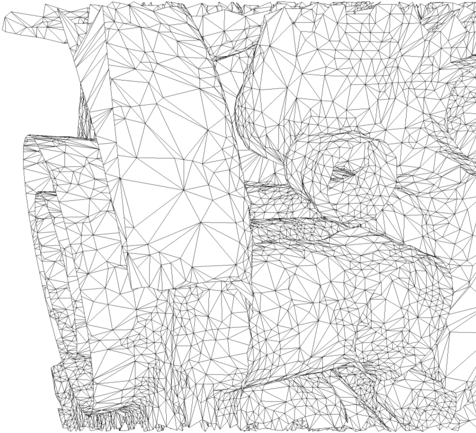
A polyhedral example





Reconstruction of surfaces



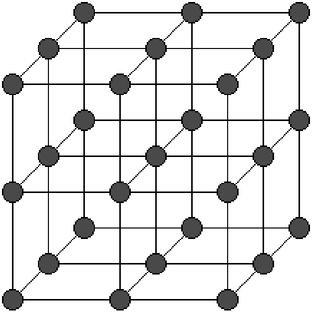


Issues

- $\mathrm{Del}_{|\mathcal{M}}(\mathsf{P})$ usually contains badly shaped simplices (of dim > 2)
- $\mathrm{Del}_{|\mathcal{M}}(\mathsf{P})$ relies on the Euclidean metric
- Curse of dimensionality

Badly-shaped simplices

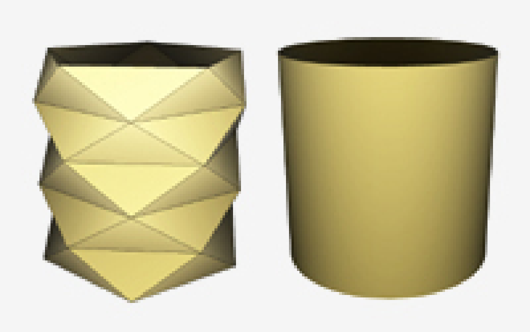
Delaunay Triangulations may contain badly-shaped i-simplices, $i > 2$, even if the vertices are well-spaced



- Each square face can be circumscribed by an empty sphere
- This remains true if the grid points are slightly perturbed therefore creating thin simplices

Badly-shaped simplices

Badly-shaped simplices lead to bad geometric approximations

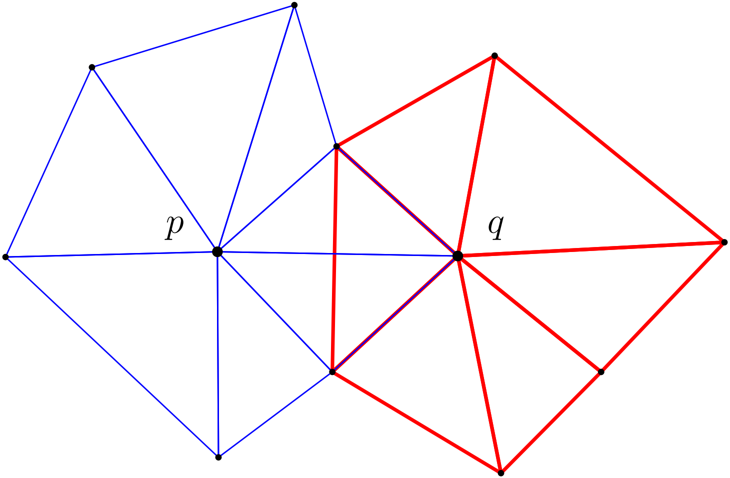


which in turn may lead to topological defects in $\mathrm{Del}_{|\mathcal{M}}(\mathsf{P})$ [Oudot]

see also [Cairns], [Whitehead], [Munkres], [Whitney]

Local Delaunay patches

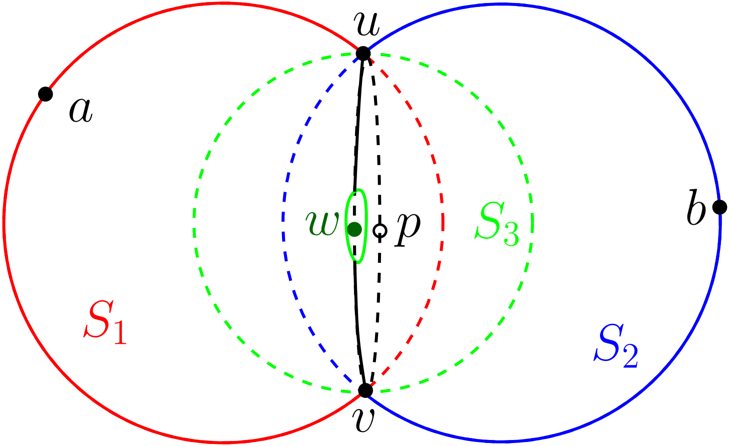
Consistency?



[B., Wormser & Yvinec 2008]; [B. & Ghosh 2010]

Intrinsic Delaunay triangulations

Existence?



[Leibon and Letscher 2000]; [Oudot 2006]; [B., Dyer and Ghosh 2012]

The curses of dimensionality

- The combinatorial complexity of DT depends exponentially on d even if the points lie on a manifold of low dimension

  The worst-case may be encountered even if the points lie on a curve, e.g. the moment curve

- The predicates required to construct DT are a polynomial of degree $d+2$ in the input coordinates

Issues

- $\mathrm{Del}_{|\mathcal{M}}(\mathsf{P})$ usually contains badly shaped simplices (of dim > 2)
- $\mathrm{Del}_{|\mathcal{M}}(\mathsf{P})$ relies on the Euclidean metric
- Curse of dimensionality

Approach

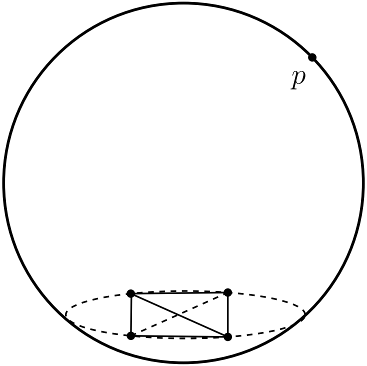
1. Introduce local versions of Delaunay triangulations
2. Make them consistent

$\Rightarrow$ Stability of DT under perturbations

Degenerate Delaunay simplices

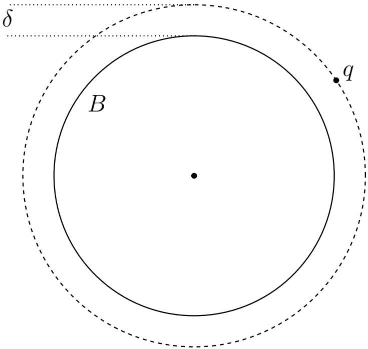
the ultimate slivers

P $\in \mathbb{R}^d$



- degenerate Delaunay j-simplex $\Longrightarrow j+1$ points on a $(j-2)$-sphere
- *always* a face of a degenerate Delaunay $(m+1)$-simplex
- $\Longrightarrow m+2$ points on the boundary of a Delaunay ball

Protection



Definition (protected)

A simplex σ is *protected* if it has a Delaunay ball B whose boundary contains no other points from P.
We say σ is δ*-protected* if $d_{\mathbb{R}^d}(q, \partial B) > \delta$ for all $q \in \mathrm{P} \setminus \sigma$.

[Funke et al. 2005]

Quantifying genericity

Definition (δ-generic)

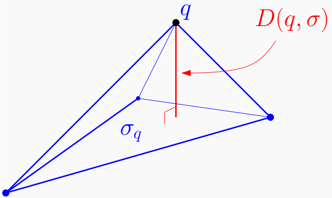
A point set $P \subset \mathbb{R}^m$ is δ-*generic* if the Delaunay m-simplices are all δ-protected.

- guarantees no degenerate m-simplices
- this definition is uninteresting without a sampling radius, ϵ
- $\nu_0^{-1} = \epsilon/\delta$ like a condition number

Quantifying simplex degeneracy

Towards a simplex quality bound

Altitudes



If σ_q, the face opposite q in σ is protected, then the *altitude* of q,

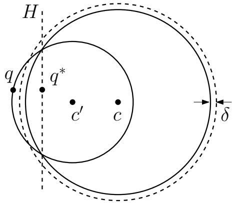
$$D(q, \sigma) = d_{\mathbb{R}^m}(q, \text{aff}(\sigma_q)),$$

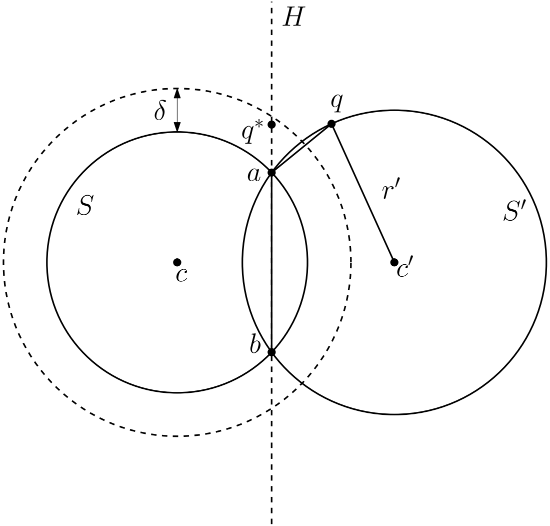
is bounded.

Assuming $\text{aff}(\mathsf{P}) = \mathbb{R}^m$, and considering only interior simplices:

- If σ^j is a Delaunay j-simplex, with $j < m$, then for any $q \in \mathsf{P} \setminus \sigma^j$, there is a Delaunay m-simplex, σ^m, with $\sigma^j \leq \sigma^m$ and $q \notin \sigma^m$.
- $\implies$ a δ-generic point set is δ-sparse: $d_{\mathbb{R}^m}(p, q) > \delta$ for all $p, q \in \mathsf{P}$

Bounding the altitudes





Using $d_{\mathbb{R}^m}(q,a) > \delta$, and $d_{\mathbb{R}^m}(a,b) > \delta$, we bound $\angle qab$.

We get $D(q,\sigma) > \frac{\sqrt{3}}{2}\frac{\delta^2}{\epsilon}$.

Thickness

The *thickness* of a j-simplex σ with diameter $\Delta(\sigma)$ is

$$\Upsilon(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p \in \sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

Result

If P is δ-generic, with $\delta = \nu_0 \epsilon$, then

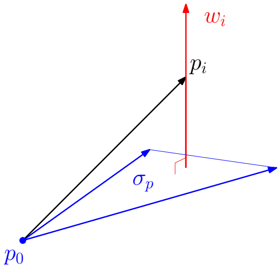
$$\Upsilon(\sigma) > \Upsilon_0 = \frac{\sqrt{3}\nu_0^2}{4},$$

for all deep interior Delaunay simplices σ.

Thickness and singular values

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & w_i & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & p_i & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}P = I$$



Lemma

Let $\sigma = [p_0, \ldots, p_j]$, and let P be the $d \times j$ matrix whose columns are $p_i - p_0$. Then $s_1(P) \leq \sqrt{j}\Delta(\sigma)$, and

$$s_j(P) \geq \sqrt{j}\Upsilon(\sigma)\Delta(\sigma).$$

Thus $\Upsilon(\sigma)^{-1} \geq \frac{s_1(P)}{s_j(P)} = \kappa(P)$, the condition number of P.

Whitney angle bound

Lemma

Let σ be a j-dimensional simplex $j \leq m$, whose vertices lie at distance $\leq \eta$ from a hyperplane $H \subset \mathbb{R}^d$. Then

$$\sin \angle(\mathrm{aff}(\sigma), H) \leq \frac{2\,\eta}{\Upsilon(\sigma)\,\Delta(\sigma)}$$

Almost circumcentres

Definition of ξ-centre

A point x is a ξ-centre for σ if
$|d_{\mathbb{R}^m}(x,p) - d_{\mathbb{R}^m}(x,q)| \leq \xi$ for all vertices $p, q \in \sigma$.

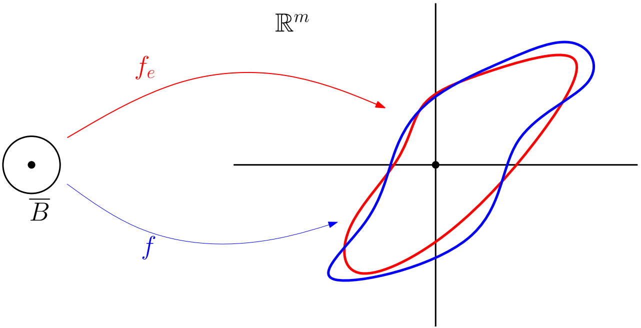
Lemma

Suppose σ is an Υ_0-thick m-simplex with $L(\sigma) \geq \mu_0 \epsilon$.
If x is a ξ-centre for σ and
$d_{\mathbb{R}^m}(x,p) < 2\epsilon$ for all $p \in \sigma$,

then

$$x \in B = B_{\mathbb{R}^m}(C(\sigma); \eta), \qquad \text{where } \eta = \frac{2\xi}{\Upsilon_0 \mu_0}.$$

Circumcentres under metric perturbation



$$f_e, f : \overline{B} \to \mathbb{R}^m$$

$$f_e : x \mapsto (d_{\mathbb{R}^m}(p_1, x) - d_{\mathbb{R}^m}(p_0, x), \ldots, d_{\mathbb{R}^m}(p_m, x) - d_{\mathbb{R}^m}(p_0, x))^{\mathsf{T}}$$

$$f : x \mapsto (d(p_1, x) - d(p_0, x), \ldots, d(p_m, x) - d(p_0, x))^{\mathsf{T}}$$

$f^{-1}(0) \neq \emptyset \implies$ there is in B a circumcentre for σ w.r.t. the metric d

[Cohen-Steiner]

Stability under metric perturbation

Theorem

Suppose P *is* δ*-generic for* P_I*, with sampling radius* ϵ *and* $\delta = \nu_0 \epsilon$*. Suppose also that* $\mathrm{conv}(\mathsf{P}) \subseteq U$*, and* $d : U \times U \to \mathbb{R}$ *is such that* $|d(x,y) - d_{\mathbb{R}^m}(x,y)| \leq \rho$ *for all* $x, y \in U$*. If*

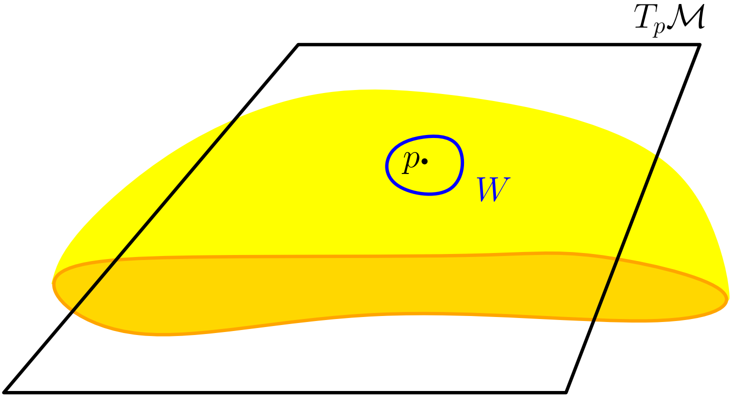
$$\rho \leq \frac{\nu_0^3}{84}\delta = \frac{\nu_0^4}{84}\epsilon,$$

then

$$\mathrm{star}(\mathsf{P}_I; \mathrm{Del}_d(\mathsf{P})) = \mathrm{star}(\mathsf{P}_I; \mathrm{Del}(\mathsf{P})).$$

Triangulating smooth submanifolds

Local Euclidean metrics



Locally project into $T_p\mathcal{M}$, which is identified with $\mathbb{R}^m$

$$\psi_p : \mathcal{M} \supset W \xrightarrow{\cong} U \subset T_p\mathcal{M}$$

$d_{\mathbb{R}^m}(\psi_p(x), \psi_p(y))$, a Euclidean metric on W

Equating restricted and intrinsic Delaunay structures

Theorem

Suppose $\mathcal{P} \subset \mathcal{M}$ is an ϵ-sample set with respect to the intrinsic metric $d_{\mathcal{M}}$, and that for each $p \in \mathcal{P}$ the set $\mathsf{P} = W \cap \mathcal{P}$ is δ-generic for $\{p\}$ with respect to the local Euclidean metric on $W = B_{\mathbb{R}^N|_{\mathcal{M}}}(p; 7\epsilon)$.
If $\delta = \nu_0 \epsilon$ and

$$\epsilon \leq \frac{\nu_0^4 \, \mathrm{rch}(\mathcal{M})}{10^5},$$

then $\mathrm{Del}_{\mathbb{R}^N|_{\mathcal{M}}}(\mathcal{P}) = \mathrm{Del}_{\mathcal{M}}(\mathcal{P})$ and they are manifold complexes.

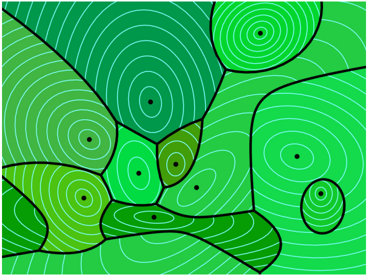
Summary

Protection parameterizes general position

- Bounded thickness for Delaunay simplices on δ-generic P
- Quantified robustness with respect to perturbations
- Sufficient conditions for the intrinsic Delaunay complex to be a triangulation (result currently relies on an embedding in $\mathbb{R}^N$)
- Genericity can be enforced by refinement

Approximating Voronoi diagrams on Riemannian manifolds

- $V(p) = \{x : d_p(x,p) \leq d_q(x,q)$ for all $p,q \in P\}$ [Labelle & Shewchuk 2003]



- $V(p) = \{x : d_x(x,p) \leq d_x(x,q)$ for all $p,q \in P\}$

[Du & Wang 2005] [Canas & Gortler 2012]

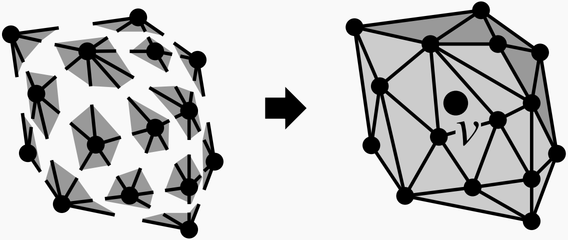
Main issue

Existence of a dual triangulation ?

Anisotropic Delaunay meshes [B., Wormser, Yvinec]

Definition

An anisotropic Delaunay mesh is a simplicial mesh in which the star of each vertex is Delaunay for the metric attached to the vertex



Uniform Anisotropic Triangulation

Take the metric cst = M_p: $d_{M_p}(x,y) = \sqrt{(x-y)^t\, M_p\, (x-y)}$

The associated Voronoi diagram $\mathrm{AVD}(\mathcal{P})$ is affine

$$\begin{aligned} d_{M_p}(x,a) < d_{M_p}(x,b) &\Leftrightarrow (x-a)^t M_p (x-a) < (x-b)^t M_p (x-b) \\ &\Leftrightarrow -2a^t M_p x + a^t M_p a < -2b^t M_p x + b^t M_p b \end{aligned}$$

Corollaries

- $\mathrm{AVD}(\mathcal{P})$ has a dual triangulation $\mathrm{Del}_{M_p}(\mathcal{P})$
- $\mathrm{Del}_{M_p}(\mathcal{P})$ can be computed efficiently

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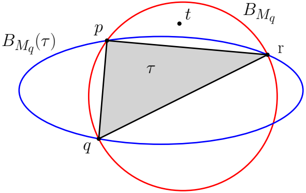
Conflicting stars



$$\tau \in \text{star}(()p) \Rightarrow t \in B_{M_p}(\tau)$$
$$\tau \notin \text{star}(()p) \Rightarrow t \notin B_{M_q}(\tau)$$

If τ is small and thick, and the metric field is Lipschitz continuous

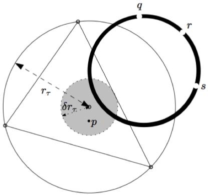
$\Rightarrow$ the vertices of the $(d+1)$-simplex $\tau * t$
are close to a $(d-1)$-dimensional M_p-sphere

Killing quasi-cospherical configurations by refinement

In order to remove element τ, we insert a new point in the picking region $\text{pick}(\tau) =$

- a small ball around the cc of τ
- minus the set of small quasi-cospherical configurations

[Li 00]



Pick_valid

Randomly pick a point p in $B(c, \delta r_\tau)$ until p does not form a quasi-cospherical configuration with any thick simplex with vertices in a neighborhood of p

Refinement algorithm

1. **Sizing field - Distortion* :**
   If $\exists v \in V$ and $\tau \in S_v$ such that $r_v(\tau) \geq r_0$ or $\gamma(\tau) \geq \gamma_0$,
   then $\texttt{Insert}(c_v(\tau))$;
2. **Shape - Radius-edge ratio :**
   If $\exists v \in V$ and $\tau \in S_v$ such that $\rho_v(\tau) > \rho_0$,
   then $\texttt{Insert}\,(\texttt{Pick\_valid}(\tau, M_v))$;
3. **Thin simplices removal :**
   If a d-simplex τ in star S_v is a sliver ($\rho_v(\tau) \leq \rho_0$, $\sigma_v(\tau) < \sigma_0^d$),
   $\texttt{Insert\_valid\_d}(\tau, M_v)$;
4. **Inconsistency :**
   If $\exists v \in V, \tau \in S_v$ that is inconsistent,
   then $\texttt{Insert}\,(\texttt{Pick\_valid}(\tau, M_v))$;

*Distortion : $\gamma(p,q) = \max(\|F_p F_q^{-1}\|_2, \|F_q F_p^{-1}\|_2)$ where $M_x = F_x^t F_x$

Main ingredients in the proof of termination

Picking Lemma

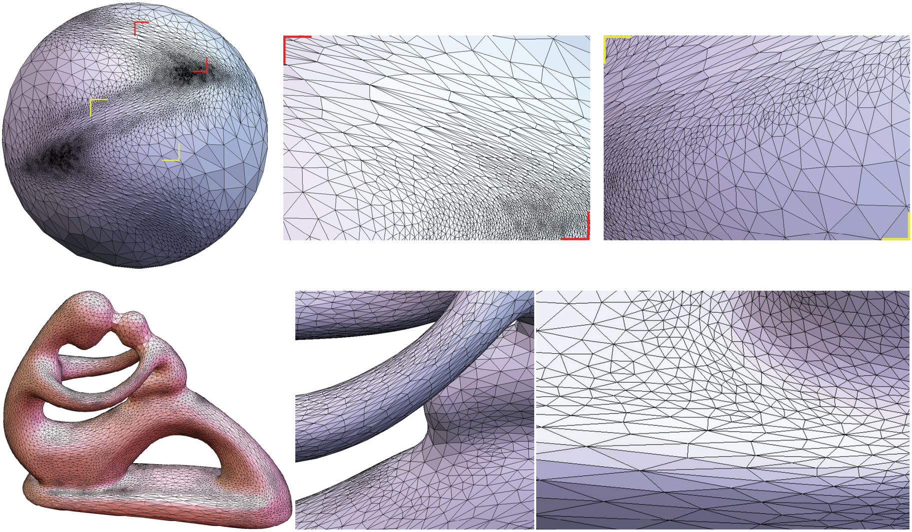
- Stability of circumcenters of thick simplices
- Volume of a forbidden region $\rightarrow 0$ when $\gamma_0 \rightarrow 0$
  (the other parameters being fixed)
- Number of forbidden regions is $O(1)$

Termination

The insertion radius is bounded from below

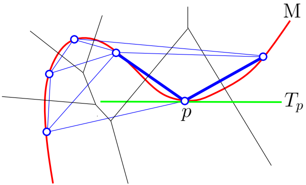
CGAL implementation

[Shi, Tournois, Yvinec]



The tangential Delaunay complex

[Freedman 2002], [B.& Flottoto 2004], [Cheng, Dey, Ramos 2005]



1. Construct the star of $p \in \mathcal{P}$ in the Delaunay triangulation $\mathrm{Del}_{Tp}(\mathcal{P})$ of $\mathcal{P}$ restricted to T_p
2. $\mathrm{Del}_{T\mathcal{M}}(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \mathrm{star}(p)$

+ $\mathrm{Del}_{T\mathcal{M}}(\mathcal{P}) \subset \mathrm{Del}(\mathcal{P})$
+ $\mathrm{star}(p)$, $\mathrm{Del}_{T_p}(\mathcal{P})$ and therefore $\mathrm{Del}_{T\mathcal{M}}(\mathcal{P})$ can be computed without computing $\mathrm{Del}(\mathcal{P})$
– $\mathrm{Del}_{T\mathcal{M}}(\mathcal{P})$ is not necessarily a triangulated manifold

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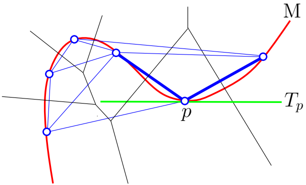

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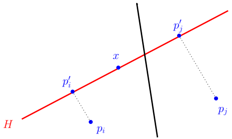
Given a k-flat H, $\mathrm{Vor}(\mathcal{P}) \cap H$ is a weighted Voronoi diagram in H



$$\|x - p_i\|^2 \leq \|x - p_j\|^2$$
$$\Leftrightarrow \quad \|x - p'_i\|^2 - \|p_i - p'_i\|^2 \leq \|x - p'_i\|^2 - \|p_j - p'_j\|^2$$

Corollary: construction of Del_{T_p}

1. project $\mathcal{P}$ onto T_p which requires $O(dn)$ time
2. construct $\mathrm{star}(p'_i)$ in $\mathrm{Del}^{\omega}(p'_i) \subset T_{p_i}$ where $\omega(p_i) = \|p_i - p'_i\|$
3. $\mathrm{star}(p_i)$ and $\mathrm{star}(p'_i)$ are isomorphic

Construction of $\mathrm{Del}_{T_p}(\mathcal{P})$

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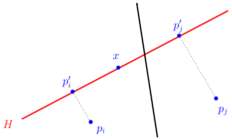

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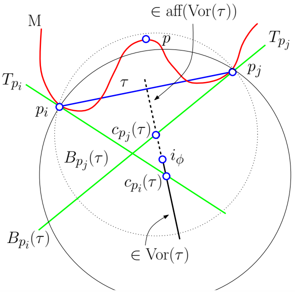
Conflicting stars in the tangential complex

1. $\tau \in \text{star}(p_i) \Rightarrow B(c_{p_i}(\tau) \cap \mathcal{P} = \emptyset$
2. $\tau \notin \text{star}(p_j) \Rightarrow B(c_{p_i}(\tau) \cap \mathcal{P} \ni p$



if τ is small and thick

$\Rightarrow$ c_i and c_j are close & $\text{aff}(\tau) \approx T_{p_i} \approx T_{p_j}$

$\Rightarrow$ $\phi := \tau * p$ is not thick

$\Rightarrow$ the vertices of ϕ are close to a $(k-1)$-sphere

Inconsistencies are due to instabilities of DT

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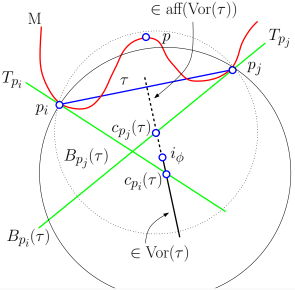

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Equating the tangential complex and the previous Delaunay structures

Theorem

Suppose $\mathcal{P} \subset \mathcal{M}$ is $(\tilde{\mu}_0\epsilon)$-sparse with respect to $d_{\mathbb{R}^N}$, and every m-simplex $\tilde{\sigma} \in \mathrm{Del}_{T\mathcal{M}}(\mathcal{P})$ is $\tilde{\Upsilon}_0$-thick, and has, for every vertex $p \in \tilde{\sigma}$, a $\check{\delta}^2$-power-protected empty ball of radius less than ϵ centred on $T_p\mathcal{M}$, with $\check{\delta} \geq \delta_0\tilde{\mu}_0\epsilon$. If $\delta_0^2\tilde{\mu}_0^2 \leq \frac{1}{7}$, and

$$\epsilon \leq \frac{\tilde{\Upsilon}_0^2\tilde{\mu}_0^3\delta_0^2 \mathit{rch}(\mathcal{M})}{1.5 \times 10^6},$$

then

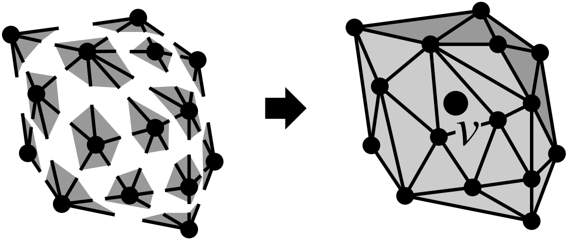
$$\mathrm{Del}_{T\mathcal{M}}(\mathcal{P}) = \mathrm{Del}_{\mathbb{R}^N|_{\mathcal{M}}}(\mathcal{P}) = \mathrm{Del}_{\mathcal{M}}(\mathcal{P}),$$

and for ϵ sufficiently small, these will be homeomorphic to $\mathcal{M}$:

$$|\mathrm{Del}_{\mathcal{M}}(\mathcal{P})| \cong \mathcal{M}.$$

Reconstruction of smooth submanifolds

1. For each vertex v, compute the star $\mathrm{star}(p)$ of p in $\mathrm{Del}_p(\mathcal{P})$
2. Remove inconsistencies among the stars by perturbing either the points or the metric (weighted DT)
3. Glue the stars to obtain a triangulation of $\mathcal{P}$



Reconstruction of smooth submanifolds

- Termination
  - If ε is small enough, the algorithm terminates and returns a tangential complex $\hat{\mathcal{M}}$ that has no inconsistent configurations
- Complexity
  - No d-dimensional data structure $\Rightarrow$ linear in d
  - exponential in k
- Approximation
  - $\hat{\mathcal{M}}$ is a PL simplicial k-manifold
  - $\hat{\mathcal{M}} \subset \text{tub}(\mathcal{M}, \varepsilon)$
  - ambient isotopic to $\mathcal{M}$

Conclusion

A quest for nice simplicial complexes

- small
- easy to compute
- precise approximations of manifolds

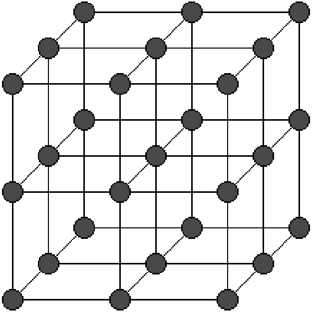
On-going work

- Witness complexes [B., Dyer, Ghosh, Oudot]
- Efficient data structures for simplicial complexes [B., Maria 2012]
- Software development

Acknowledgments

- My co-authors
  A. Ghosh, R. Dyer, S. Oudot, M. Yvinec, C. Wormser
- The CG-Learning project http://cgl.uni-jena.de/Home/WebHome
- The CGAL project http://cgal.org

Flat simplices may exist in higher dimensional DT



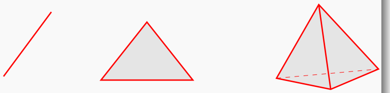
- Each square face can be circumscribed by an empty sphere
- This remains true if the grid points are slightly perturbed therefore creating flat tetrahedra

Simplicial complexes

k-simplex

The convex hull of $k+1$ points that are affinely independent

1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron



Simplicial complex

A finite collection of simplices C called the faces of C such that

- $f \in C, f \subset g \Rightarrow g \in C$
- $\forall f, g \in C$, either $f \cap g = \emptyset$ or $f \cap g \in C$

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Triangulation of differentiable manifolds

Cairns, Whitehead, Whitney [1934-1957]

Triangulation of a topological space $\mathbb{M}$

a simplicial complex $K+$ homeomorphism : $\cup K \to \mathbb{M}$

" Mr Poincaré a laissé à ses successeurs le problème de la triangulation. Il aurait été bien plus gentil de la part de Mr Poincaré de résoudre ce problème fondamental et de nous laisser quelques uns des problèmes plus amusants dont il s'est occupé."

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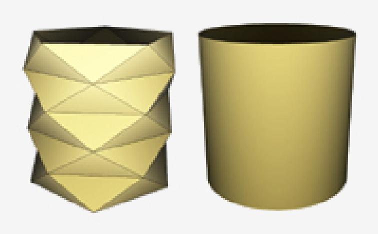
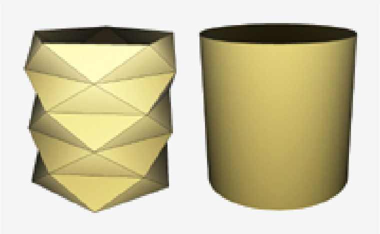
Emergence of the notion of good/thick/fat simplices



Un souci de simplicité

[Cairns, 1960] Quoique je trouve la triangulation de Whitney la plus simple jusqu'ici, je crois bien que la triangulation la plus simple reste à trouver

Emergence of the notion of good/thick/fat simplices



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