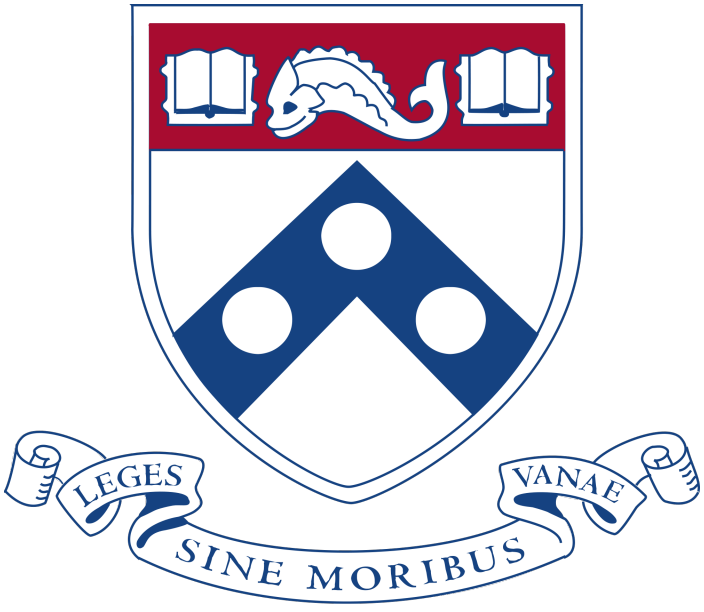
Cosheaves and Dualities in Generalized Sensor Networks

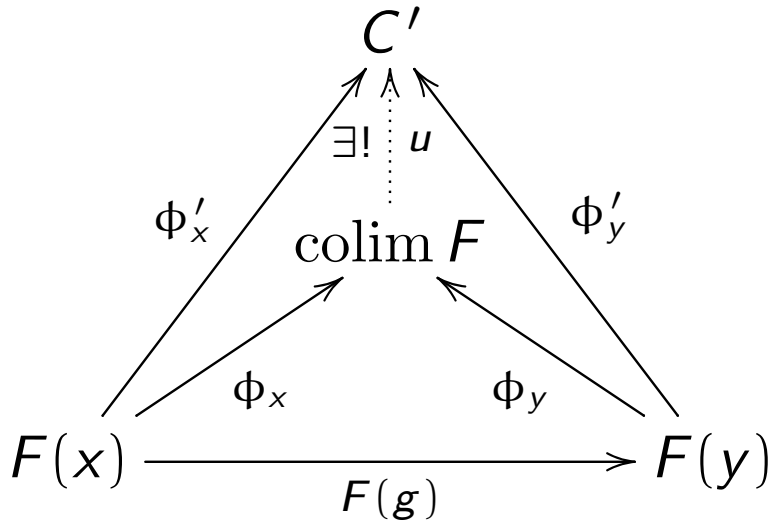


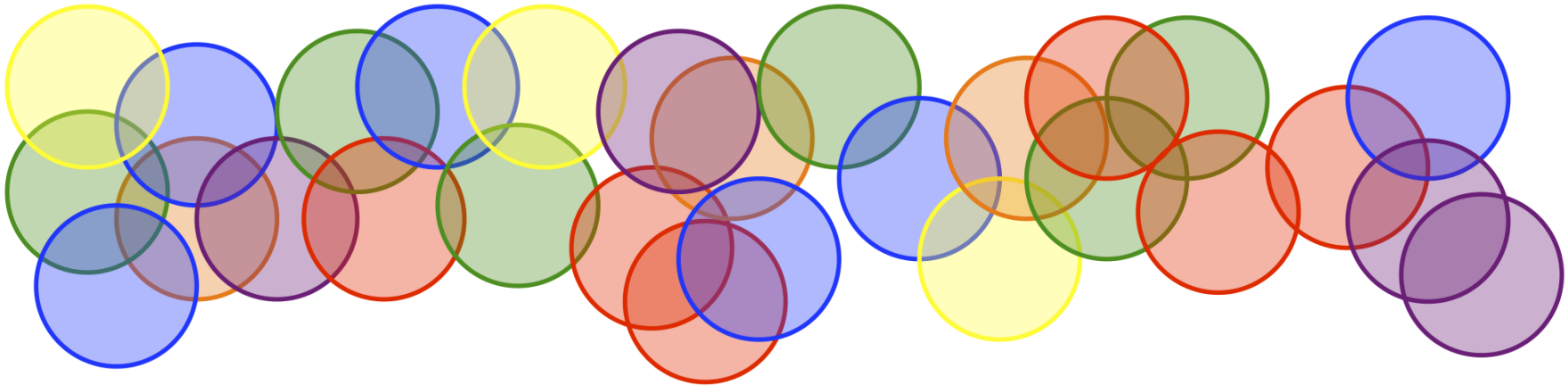
Justin Curry

University of Pennsylvania

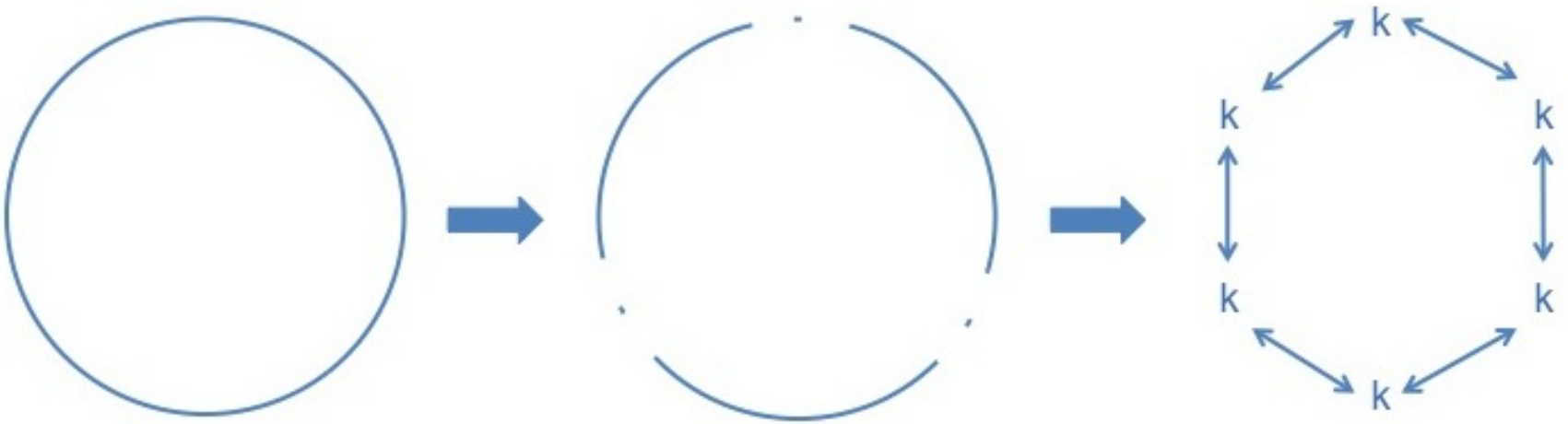
July 6, 2012

COSHEAVES





What is Cellular (Co)Homology?



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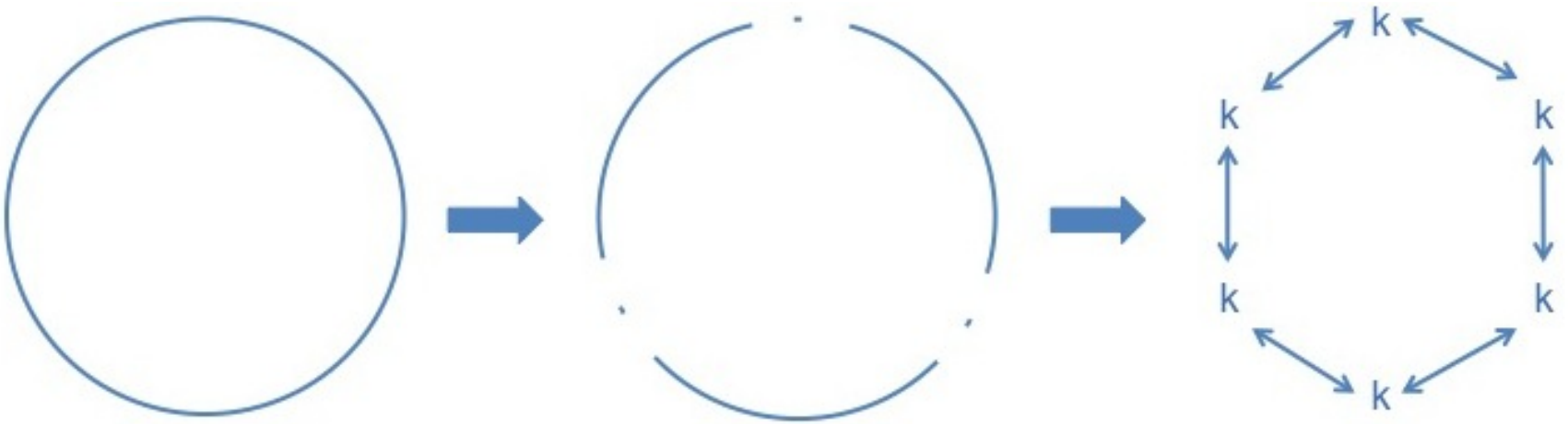

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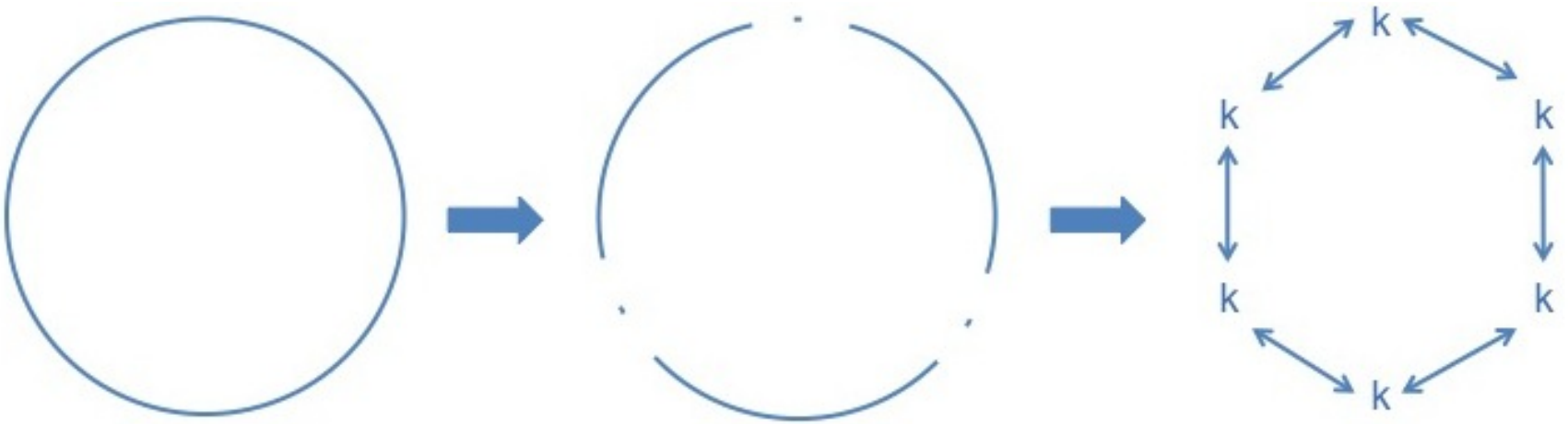

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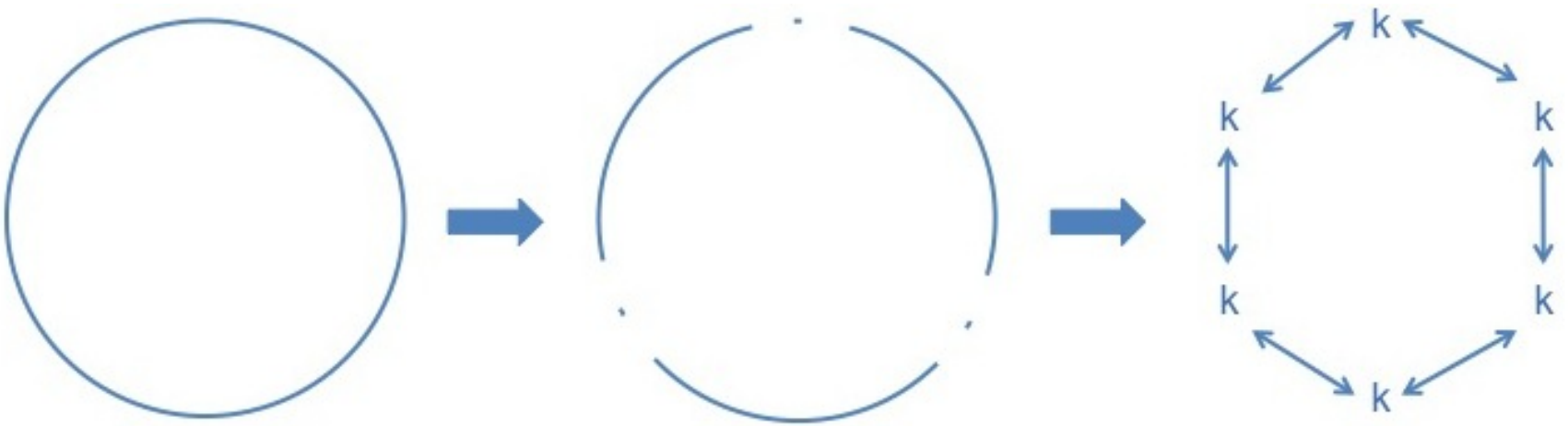

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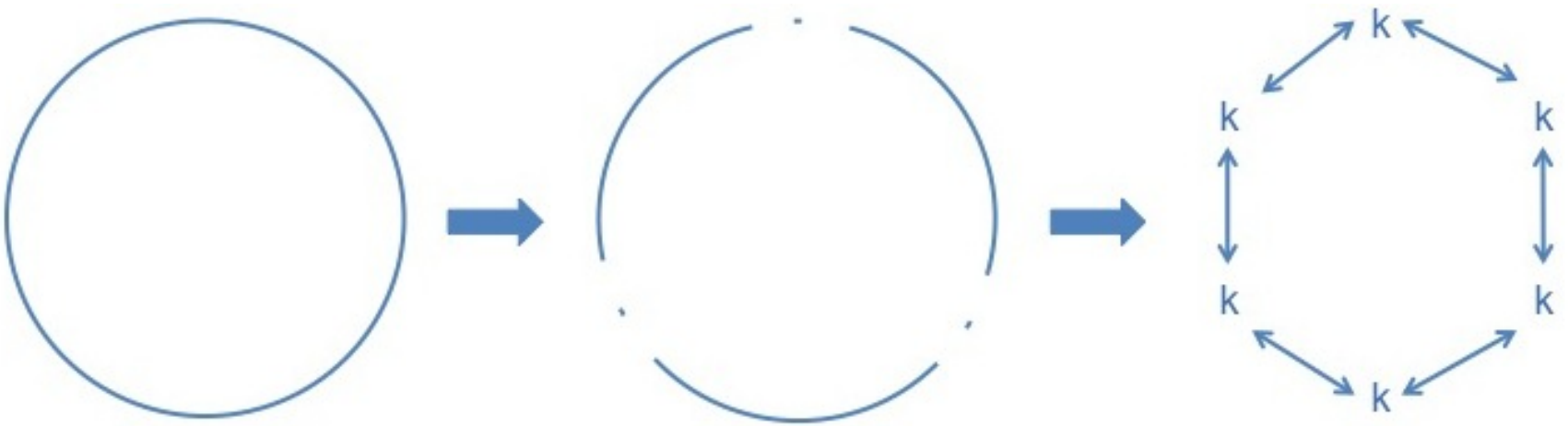

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  - Turn this into a diagram of vector spaces and compute things with linear algebra

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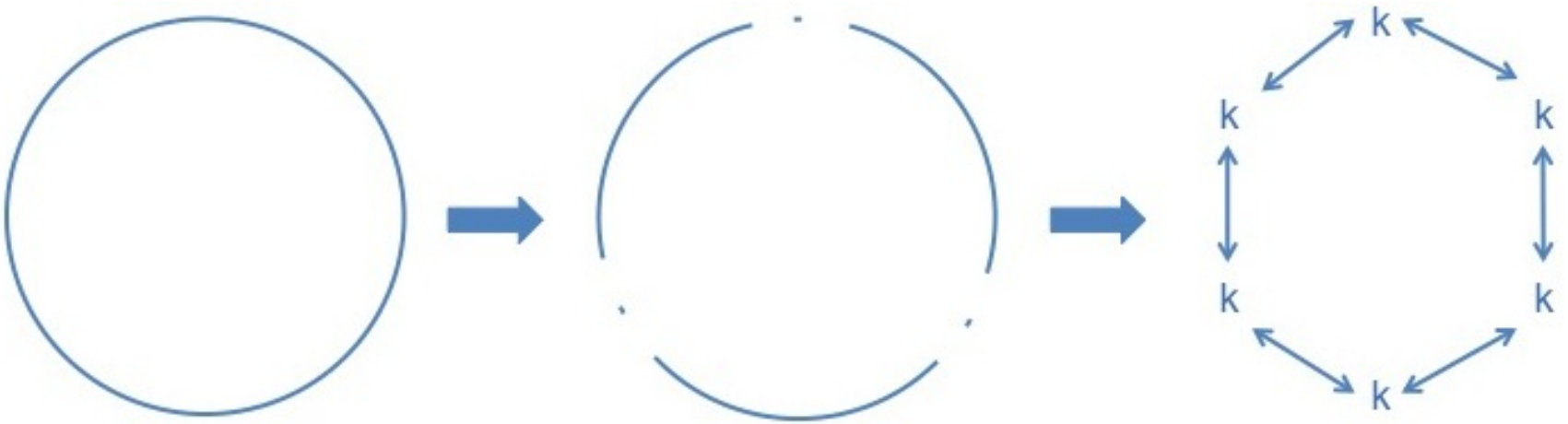

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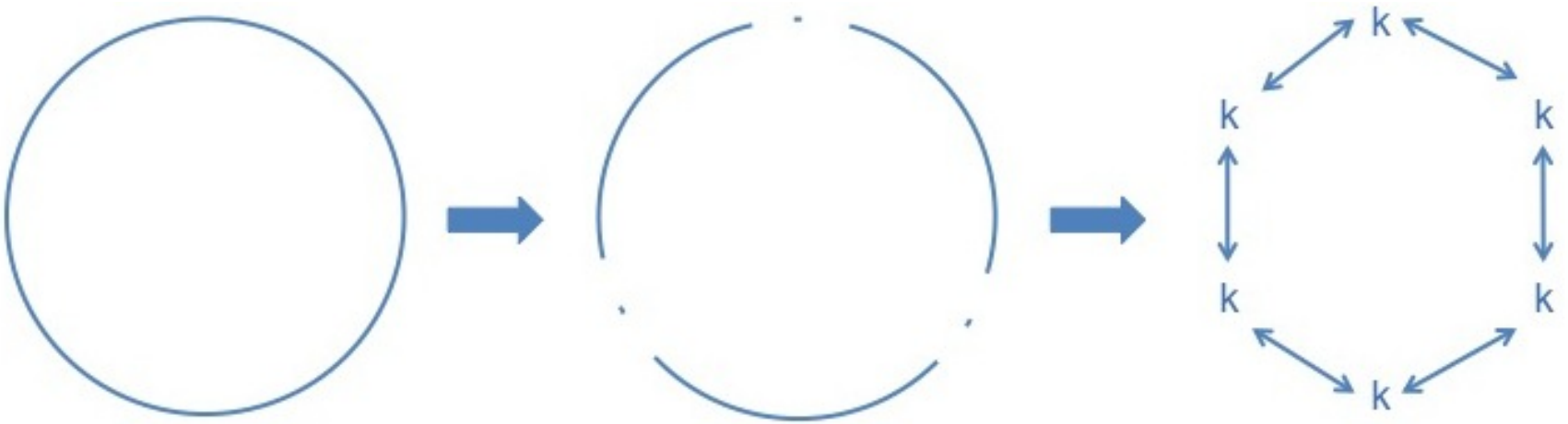

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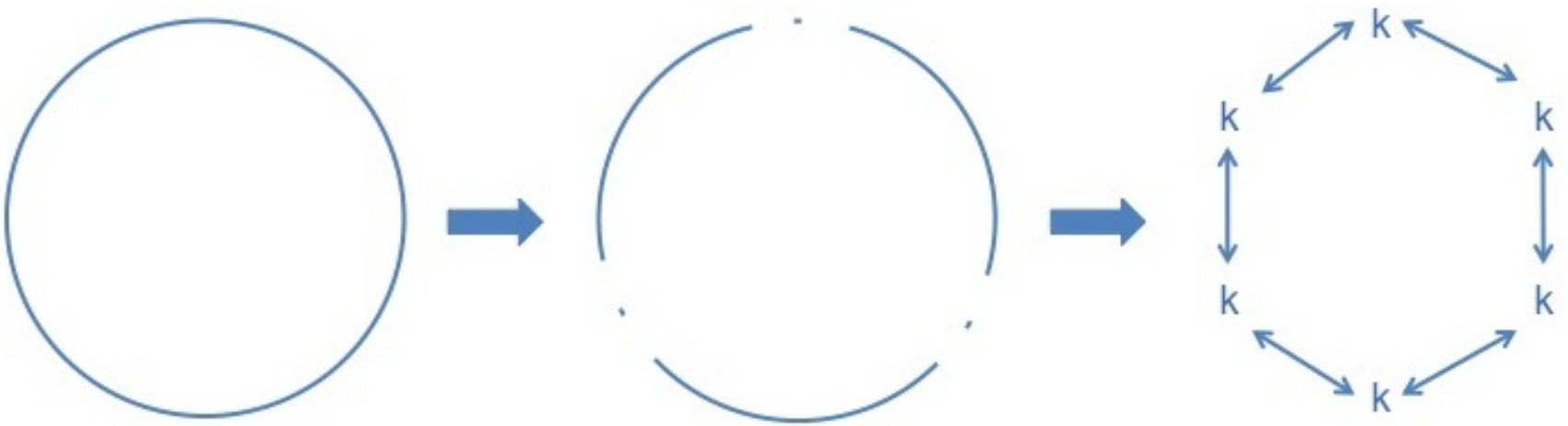

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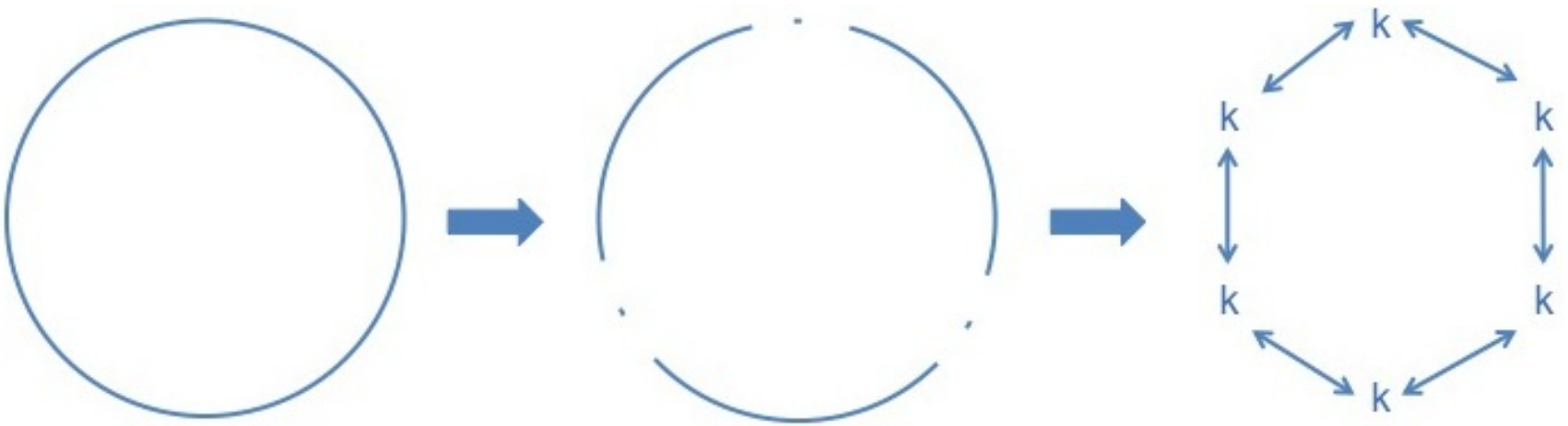

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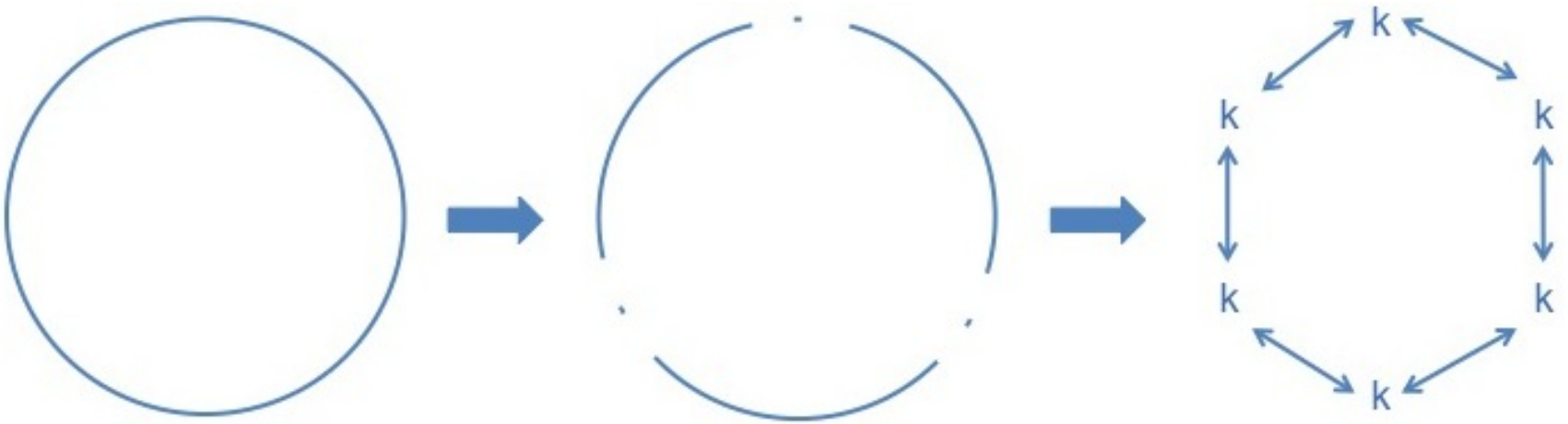

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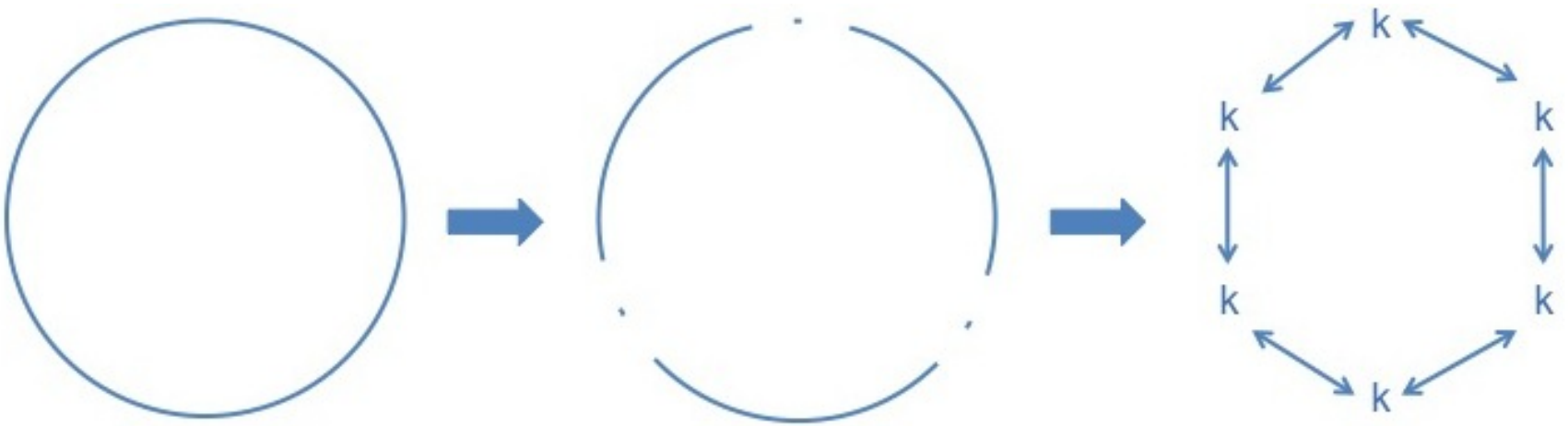

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  - (Co)Homology is the study of specific **representations** of X, i.e. *constant functors* $F : X \to \mathbf{Vect}$ or $\hat{F} : X^{op} \to \mathbf{Vect}$.

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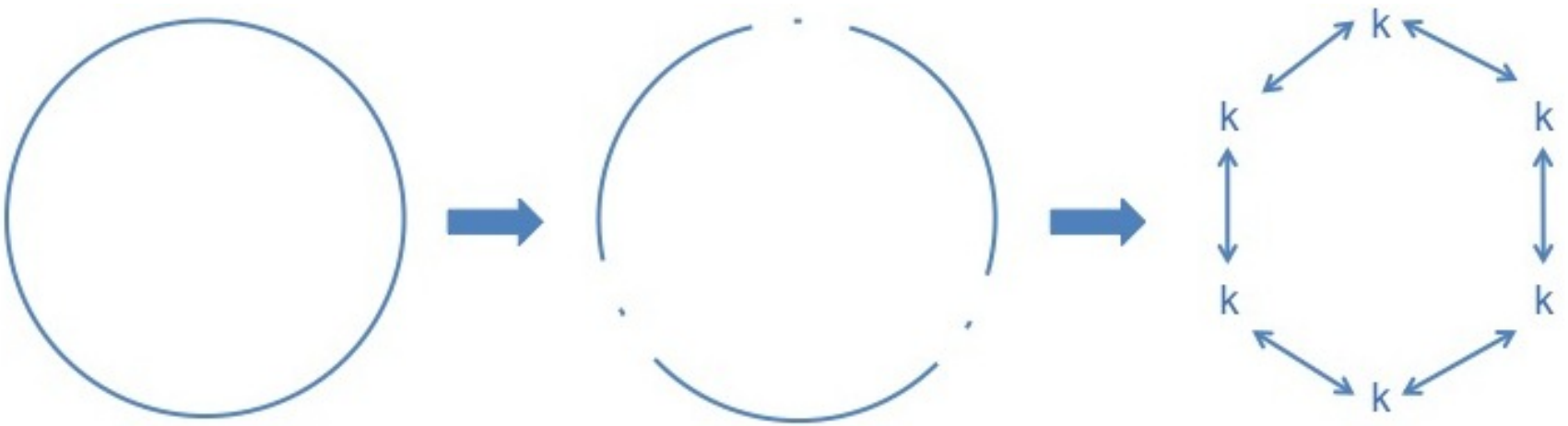

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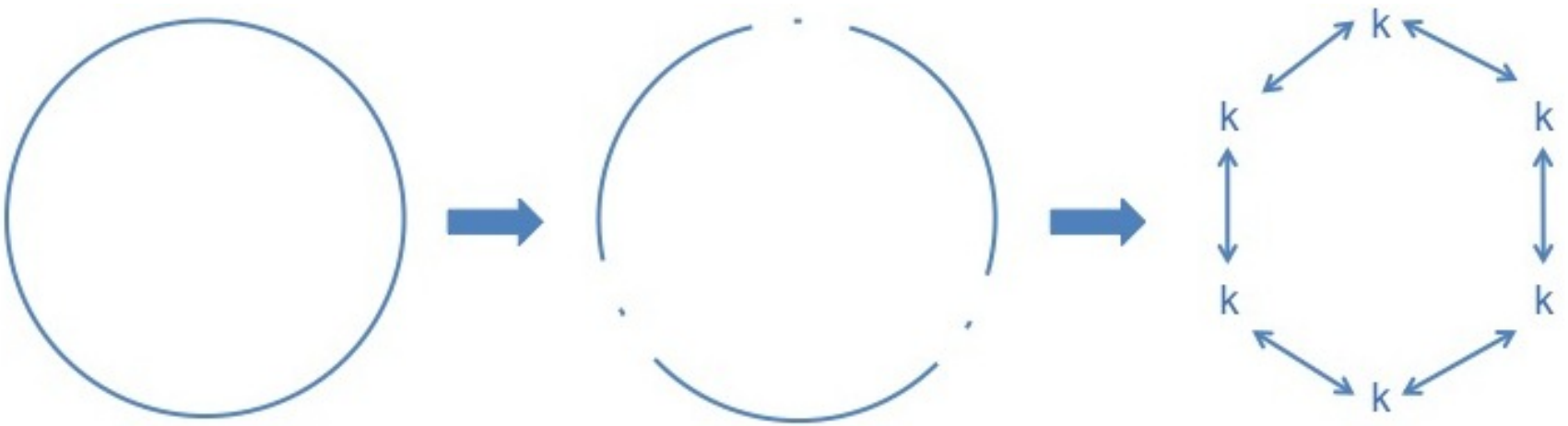

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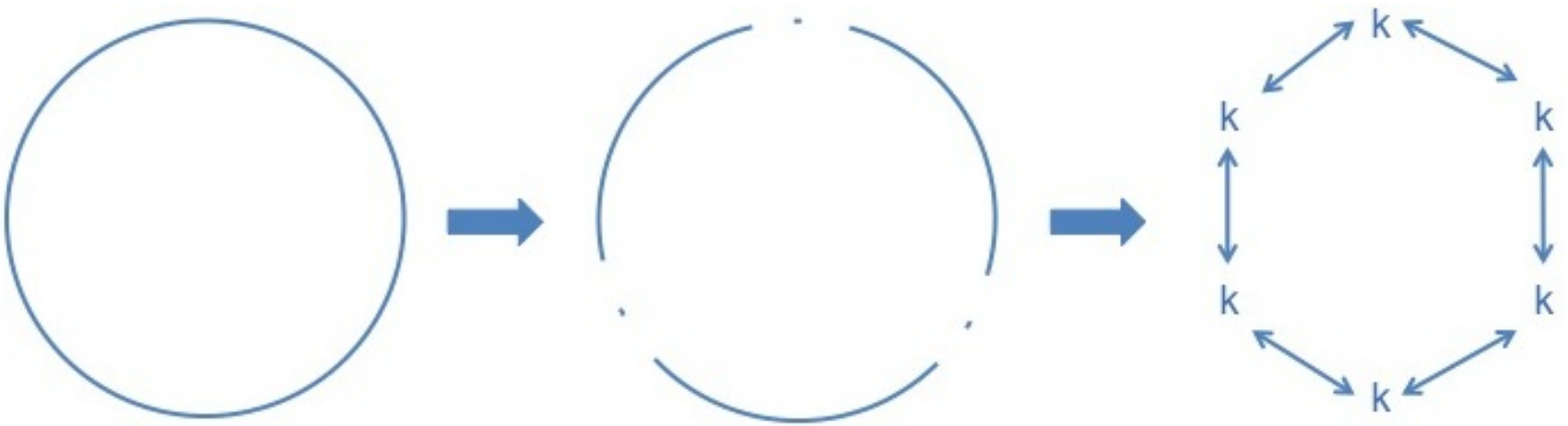

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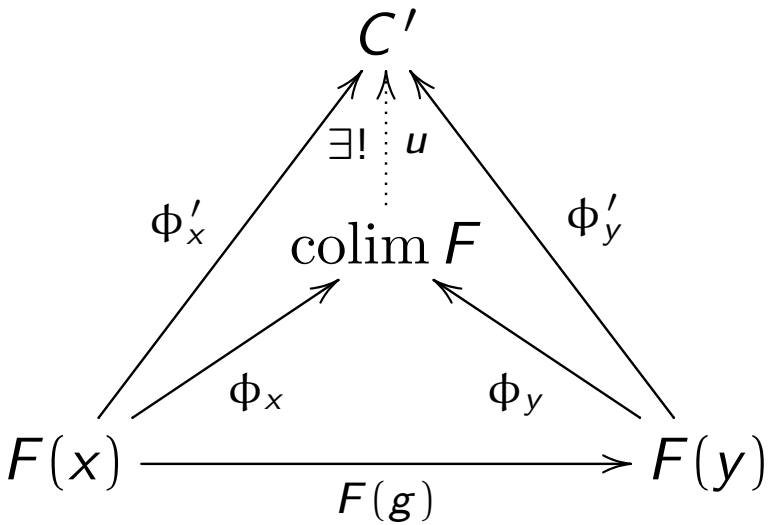

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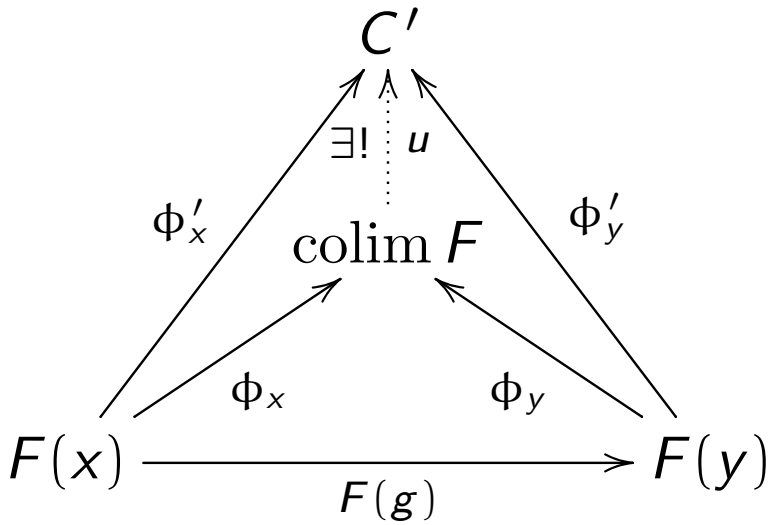

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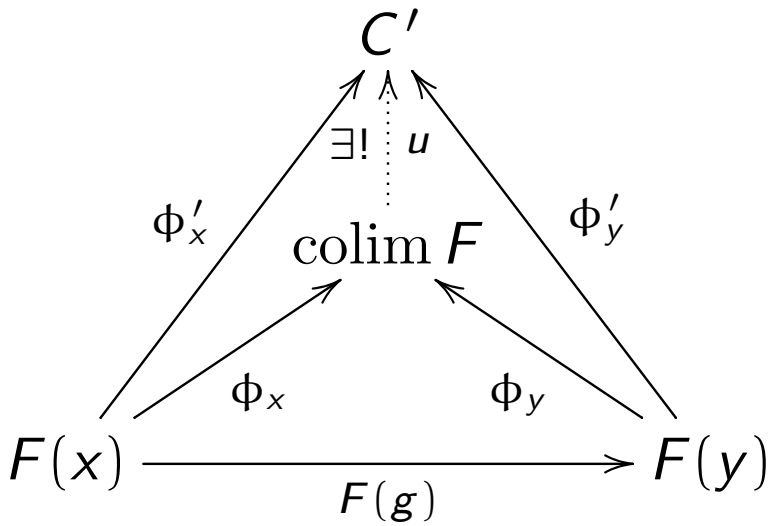

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Example (Compactly Supported Functions)

There is a sheaf

$$U \rightsquigarrow \{f : U \to \mathbb{R}\}$$

restriction maps are restriction of domain of definition.
There is a *cosheaf*

$$U \rightsquigarrow \{f : U \to \mathbb{R} \,|\, f \,\text{compact support}\}$$

extension maps are extension by zero. This is *not a sheaf.*

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  or for every cover

$$\{U_i\} \to U \qquad \operatorname{colim} \hat{\mathsf{F}}(U_i) \cong \hat{\mathsf{F}}(U)$$

Example

To a map $f : Y \to X$ can define a cosheaf of spaces on X, i.e.

$$\begin{aligned} U &\rightsquigarrow f^{-1}(U) \\ U \cup V &\rightsquigarrow f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \\ \widehat{\Gamma}(X; \hat{\mathsf{F}}) &= Y \end{aligned}$$

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To a finite poset $(X, \leqslant)$, define a topology

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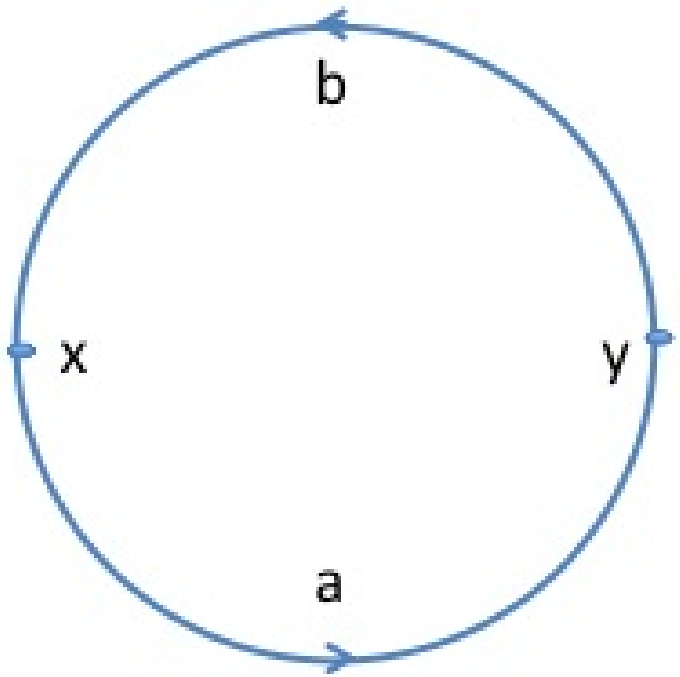
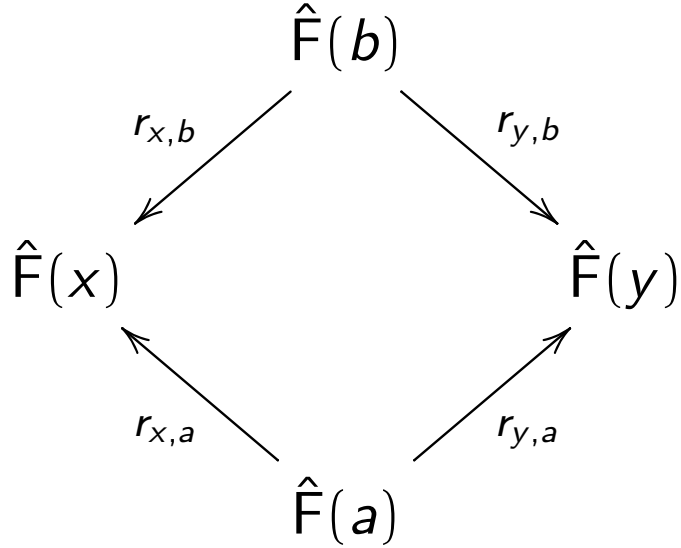
Claim (Diagrams are (Co)Sheaves)

Any functor $F : X^{op} \to \mathbf{Vect}$ defines a cosheaf and vice versa.

Cellular Cosheaves on the Circle





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- Due to presence of enough injectives/projectives for cellular sheaves and cosheaves can compute homology of sheaves and cohomology of cosheaves. *But no nice formulas.* (Aside from simplicial replacement.)

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- In the case where X is a cell structure on a manifold and F is a cell sheaf, then observe that F defines a cell cosheaf on the dual triangulation $\widehat{X}$ and *all invariants agree*

$$\begin{array}{ccc} F(\sigma^i) & \xrightarrow{\rho_{\sigma,\tau}} & F(\tau^{i+1}) \\ \downarrow & & \downarrow \\ F(\hat{\sigma}^{n-i}) & \xrightarrow{\rho_{\sigma,\tau}} & F(\hat{\tau}^{n-i-1}) \end{array}$$

Dualities between Sheaves and Cosheaves

- $\boldsymbol{vect}$ - the category of finite dimensional vector spaces has obvious anti-involution $\star : \boldsymbol{vect}^{op} \to \boldsymbol{vect}$ given by $V \rightsquigarrow V^*$
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- **Curry-Lipsky '12, Schneider '98** There is a bona fide derived equivalence

$$\widehat{\mathcal{P}} : D^b(\boldsymbol{Shv}) \leftrightarrow D^b(\boldsymbol{CoShv}) : \mathcal{P}$$

with $\widehat{\mathcal{P}} \dashv \mathcal{P}$. Moreover, $\widehat{\mathcal{P}}(\omega^\bullet_X) = \hat{\mathbf{k}}_X$ and $\mathcal{P}(\hat{\mathbf{k}}_X) = \omega^\bullet_X$.

- Poincaré-Verdier duality is an exchange between sheaves and cosheaves

$$\mathbf{Hom}_{D^b(\boldsymbol{Shv})}(F, \omega^\bullet_X) \cong \mathbf{Hom}_{D^b(\boldsymbol{CoShv})}(\widehat{\mathcal{P}}(F), \hat{\mathbf{k}}_X)$$

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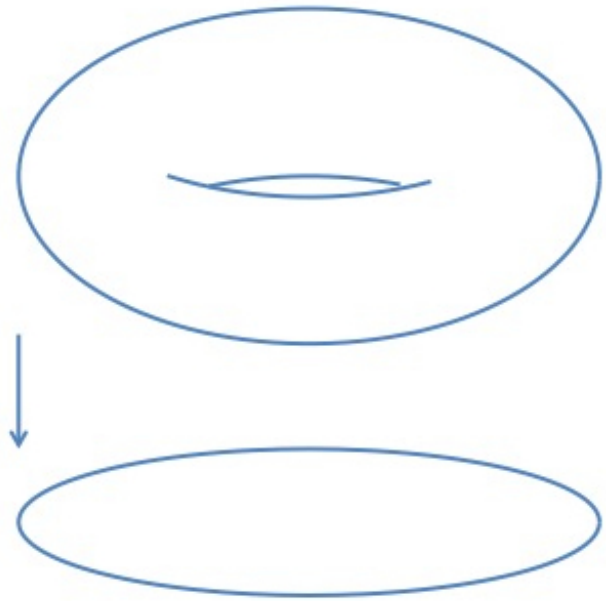

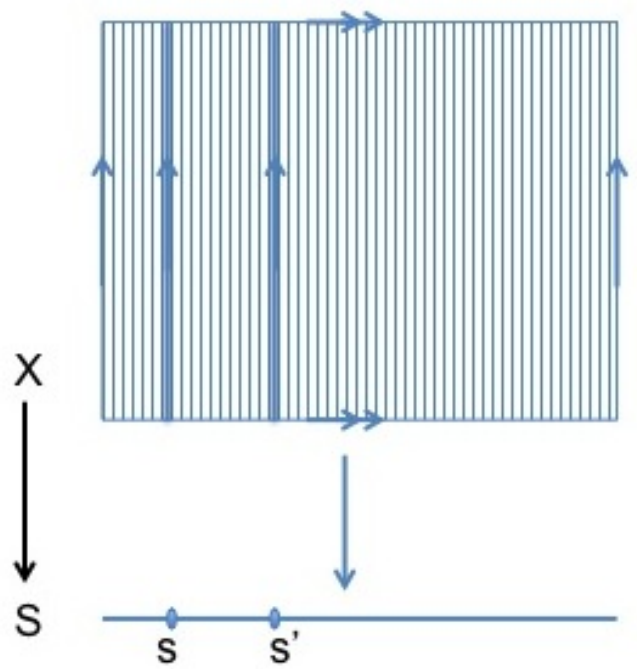

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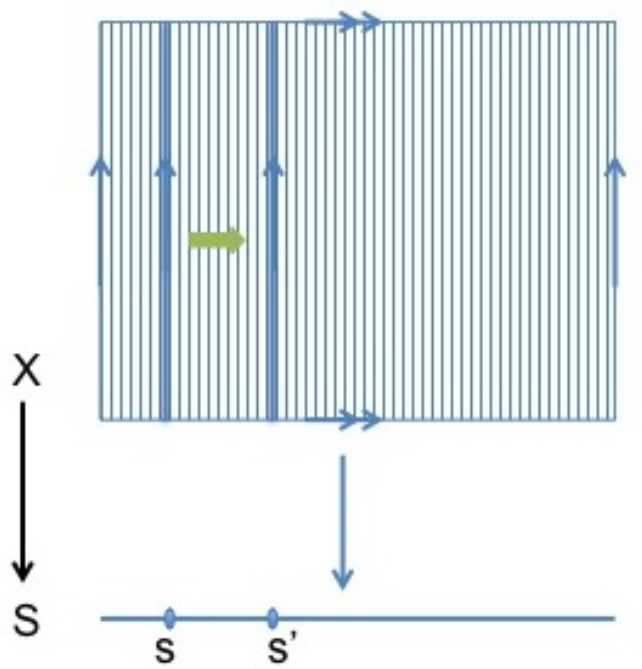

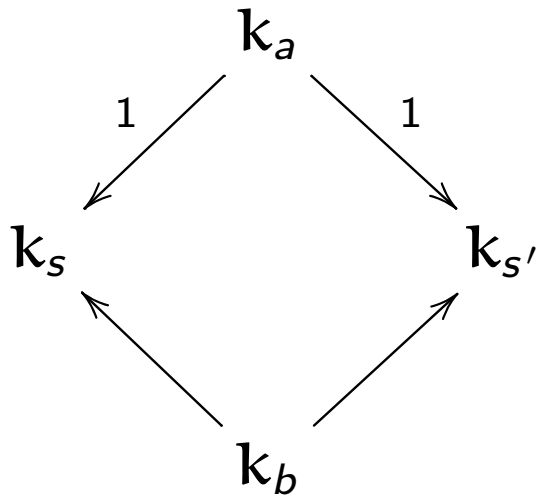

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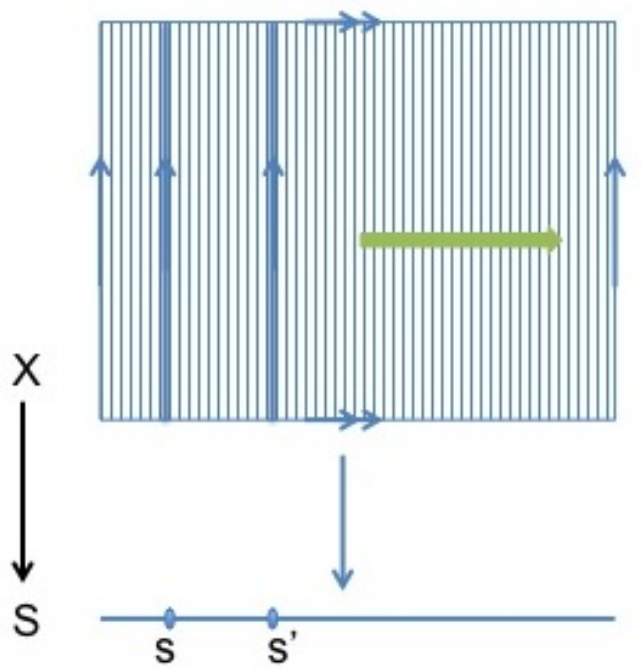

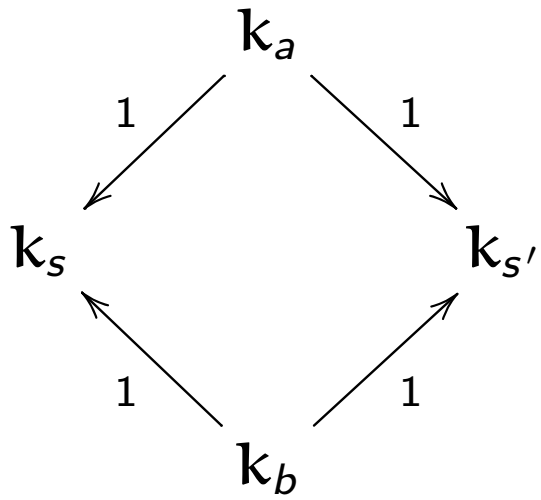

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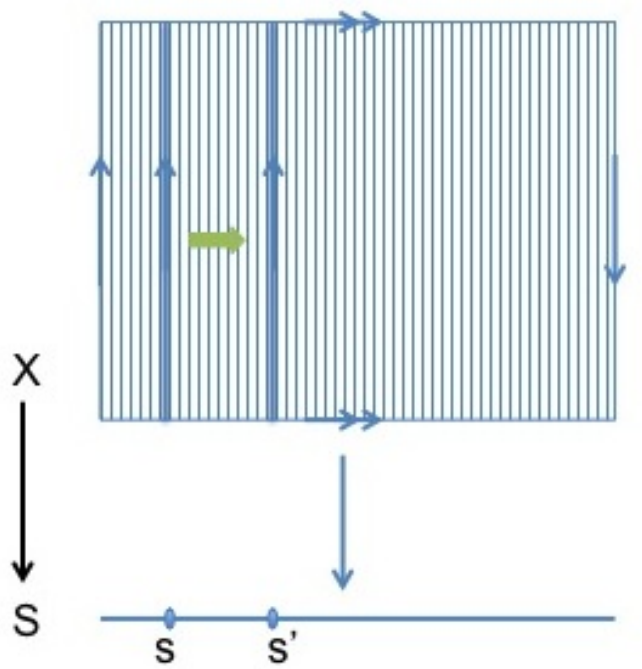

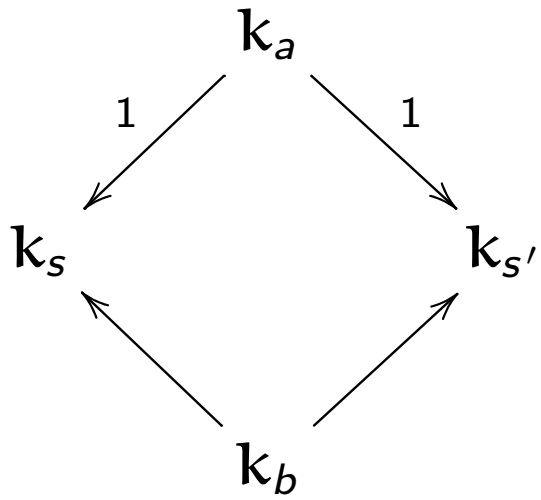

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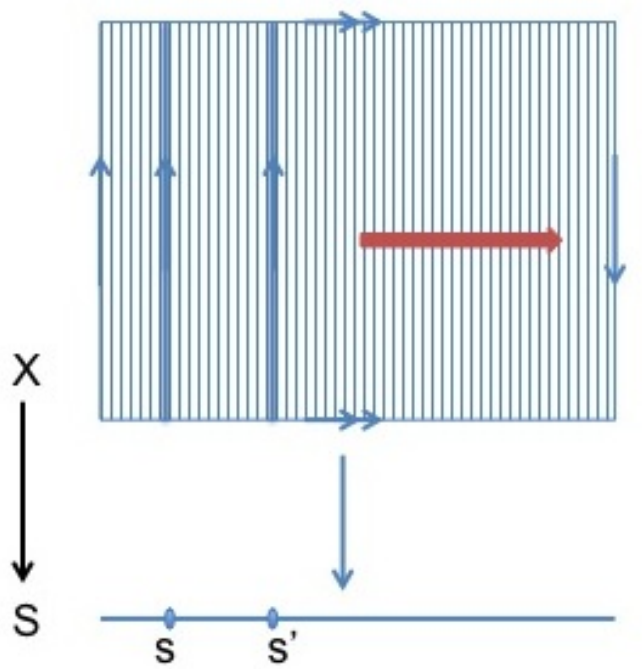

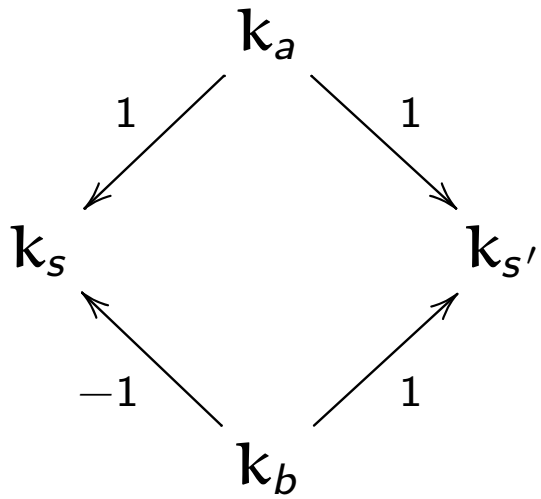

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- **Thom** tells us that all sub or semi-analytic maps of analytic varieties are stratifiable.
- **MacPherson** observed that any stratified mapping $\pi : X \to S$ and any homology functor $H_i(-; k)$ together produces a constructible cosheaf whose costalks are $H_i(\pi^{-1}(s); \mathbf{k})$

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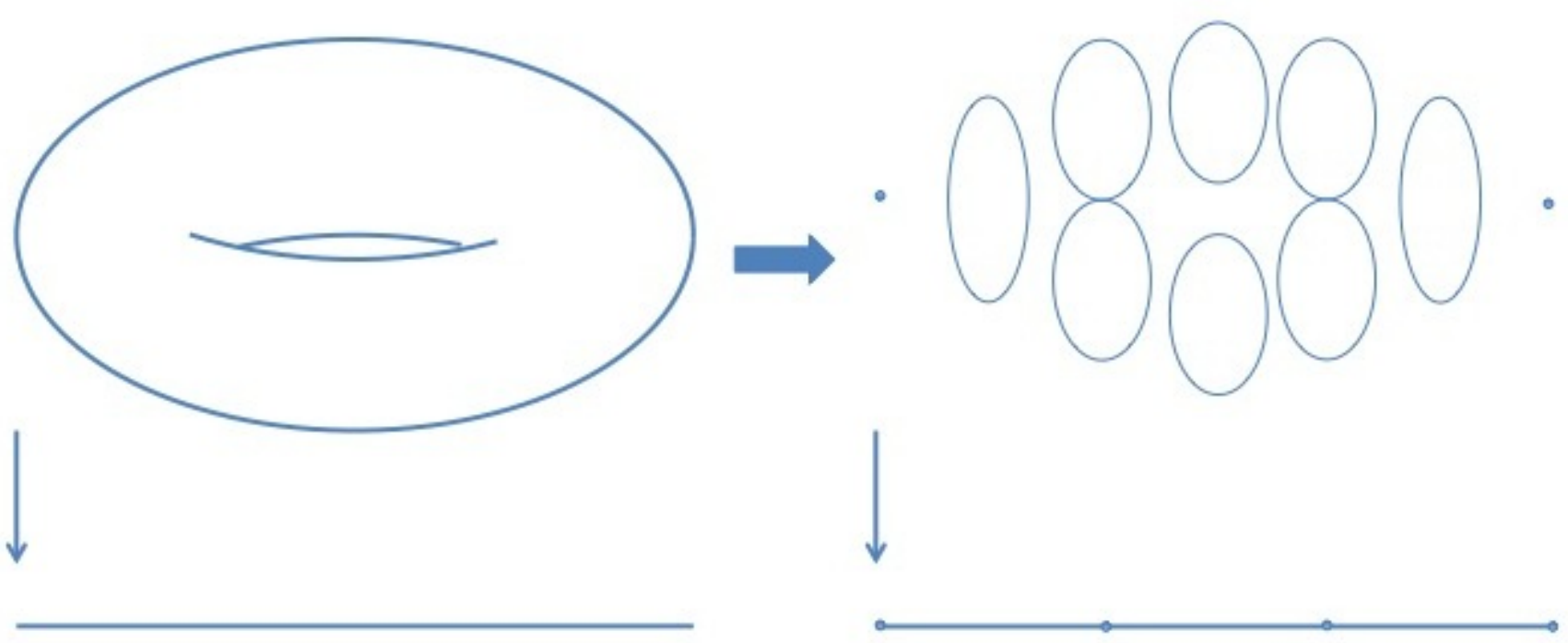

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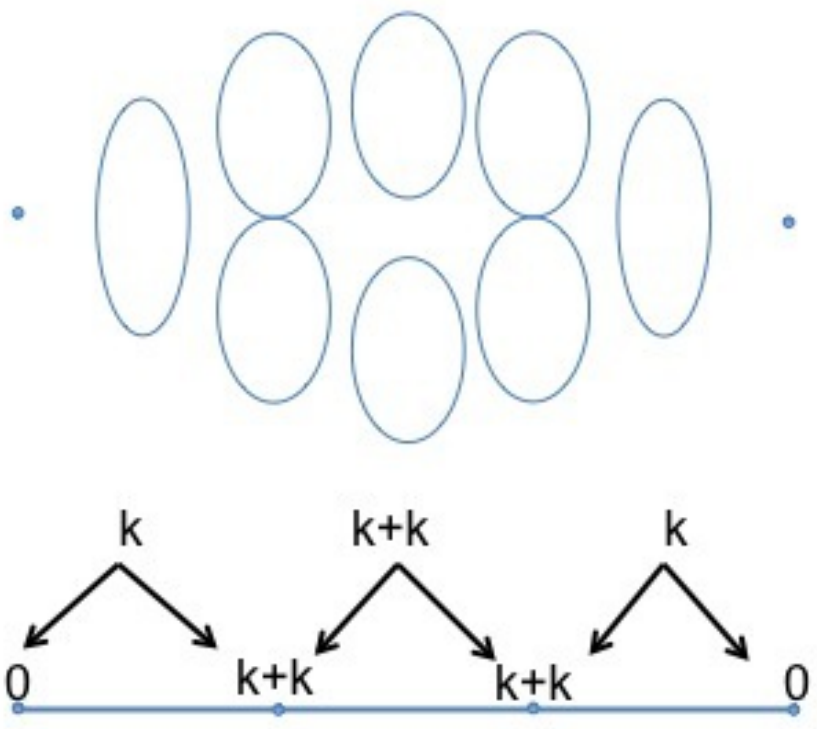

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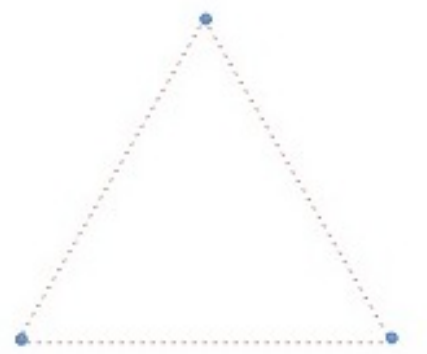

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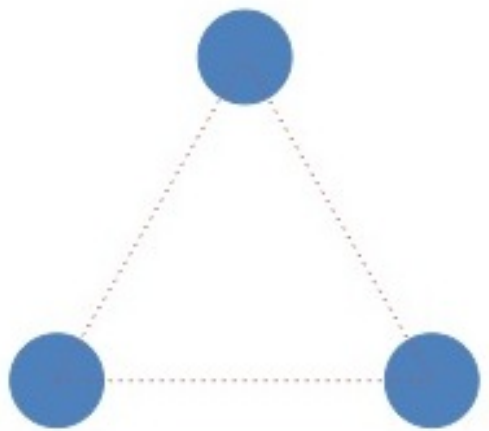

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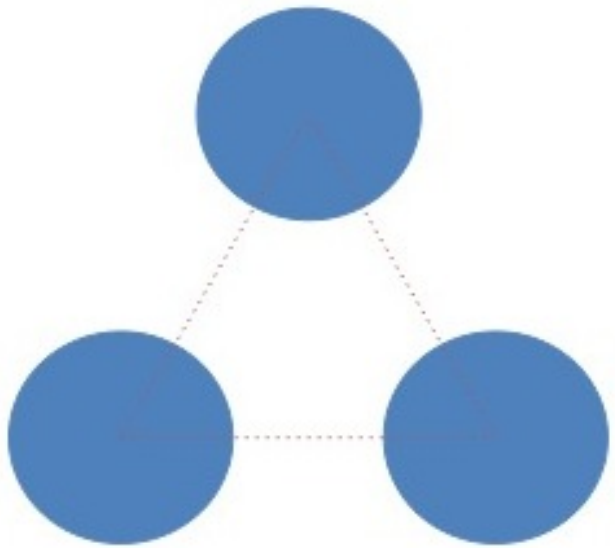

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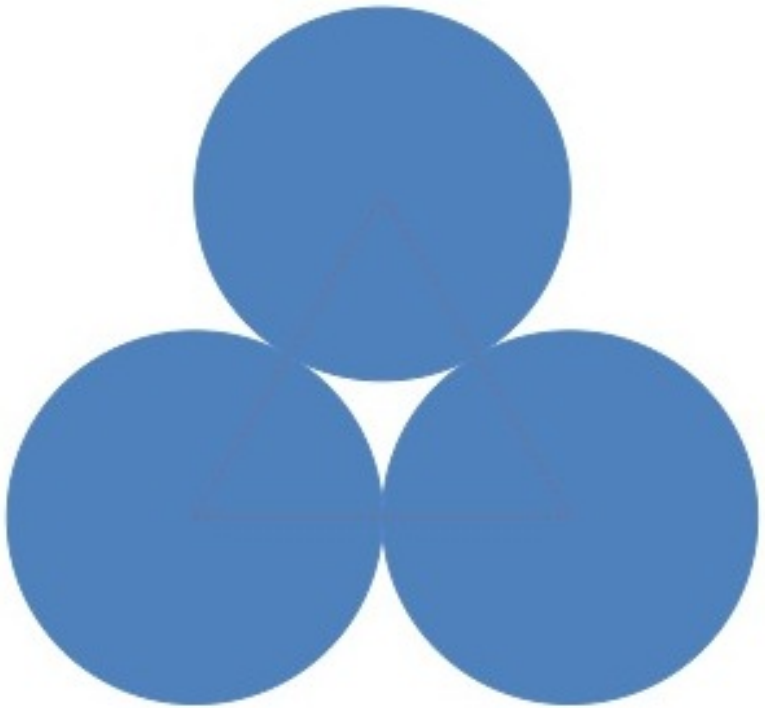

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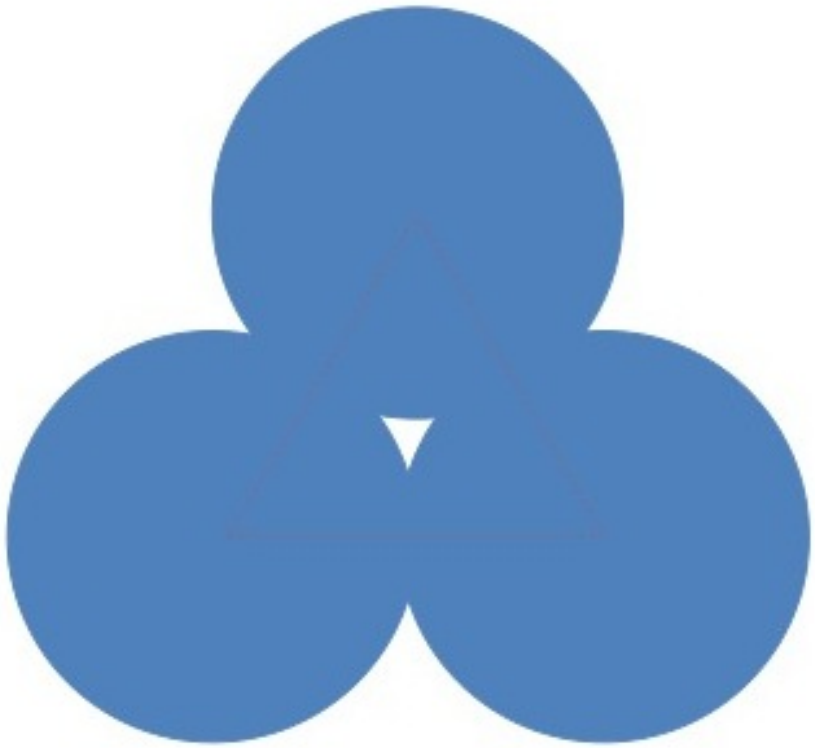

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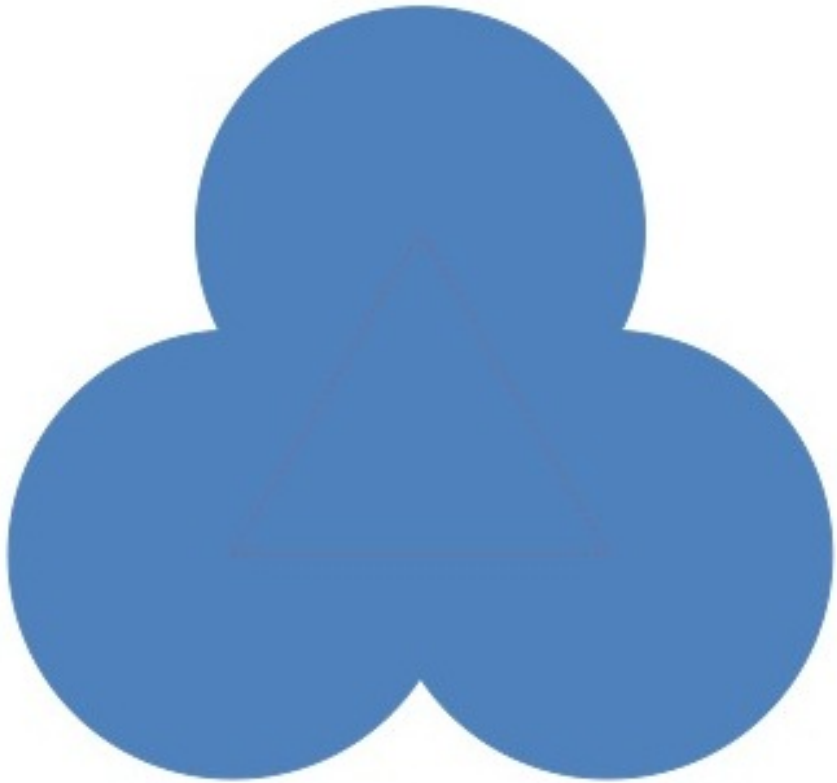

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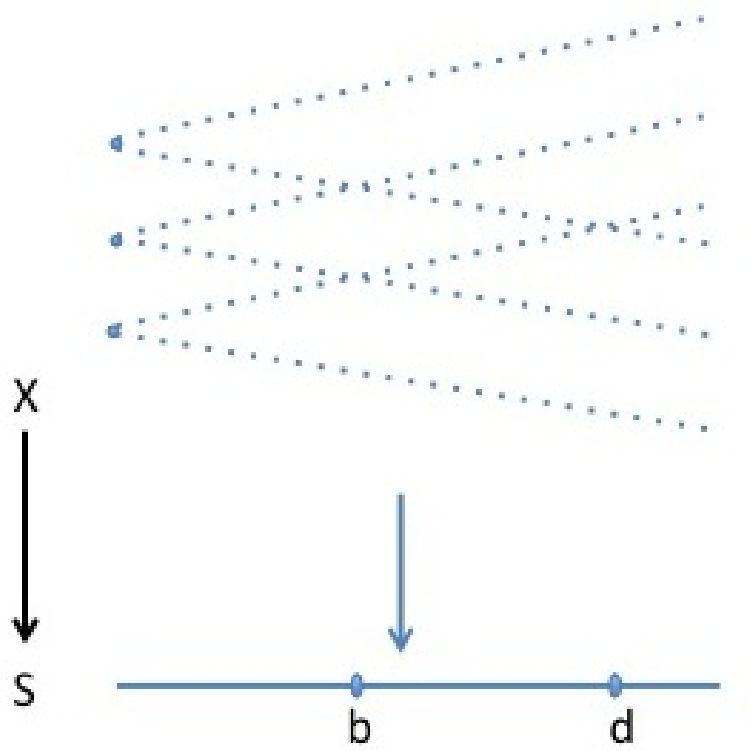

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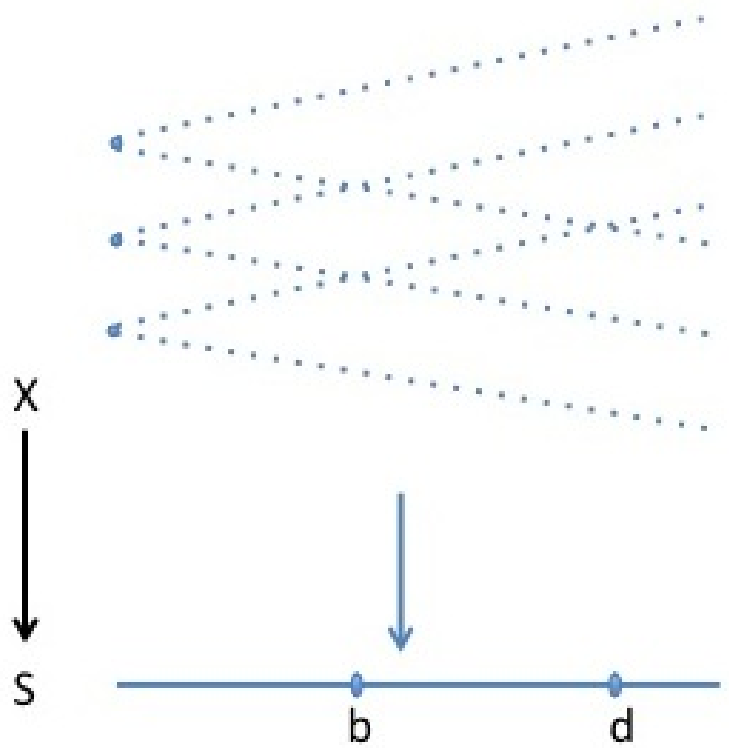

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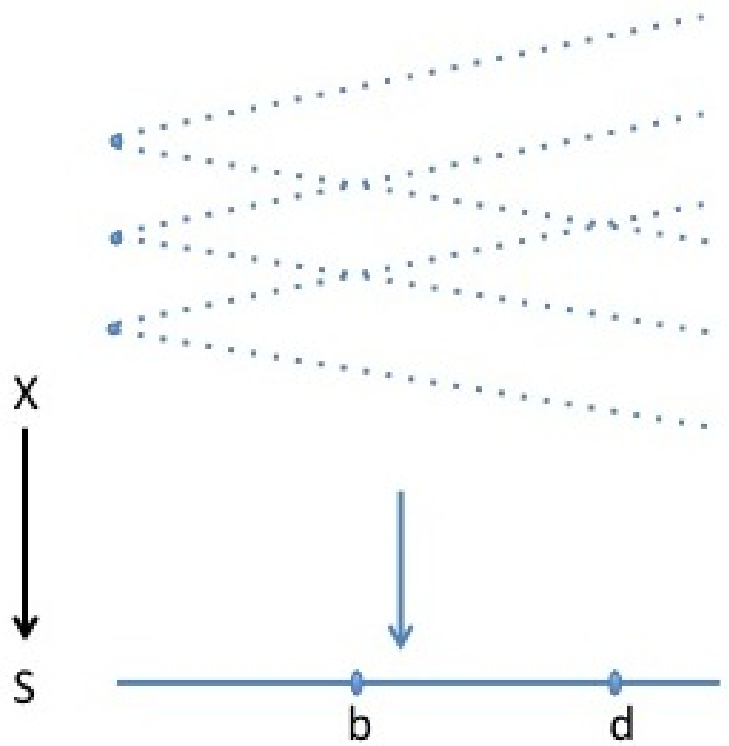

- **Main Point:** Use

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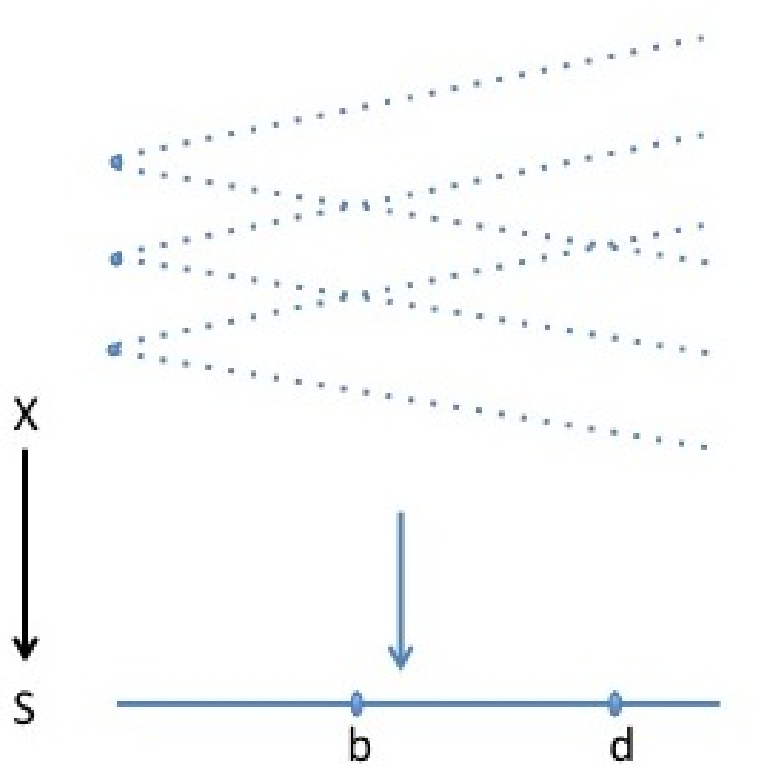

- **Main Point:** Barcodes correspond to the Atiyah-Remak decomposition of the persistence cosheaf (cf. Carlsson-de Silva Zig-Zag)

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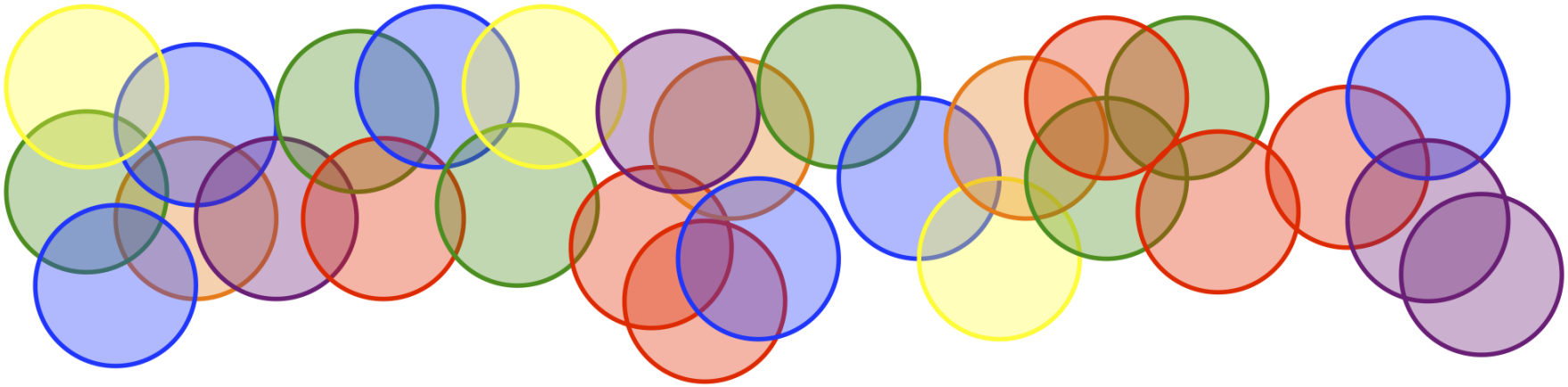

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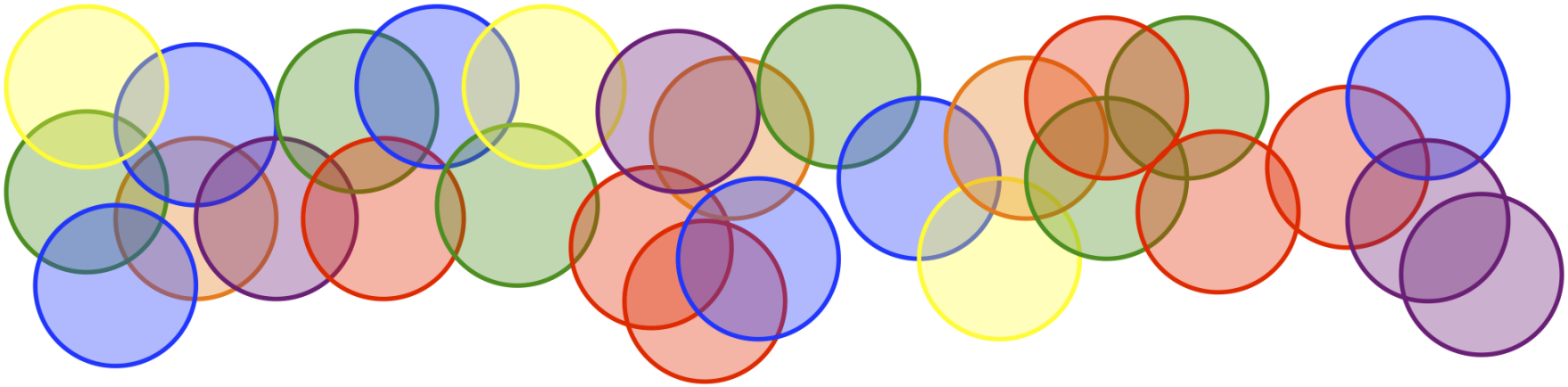

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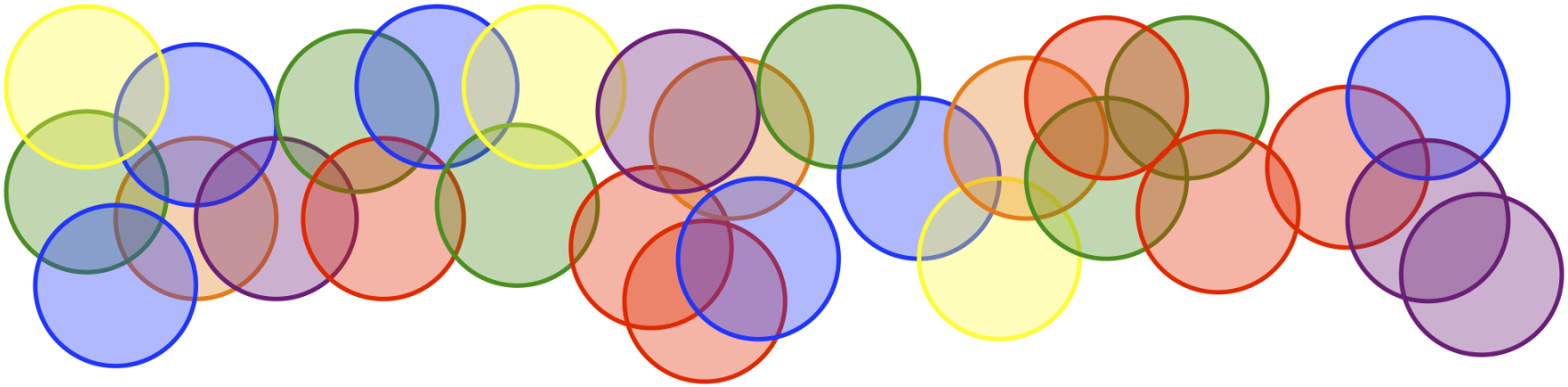

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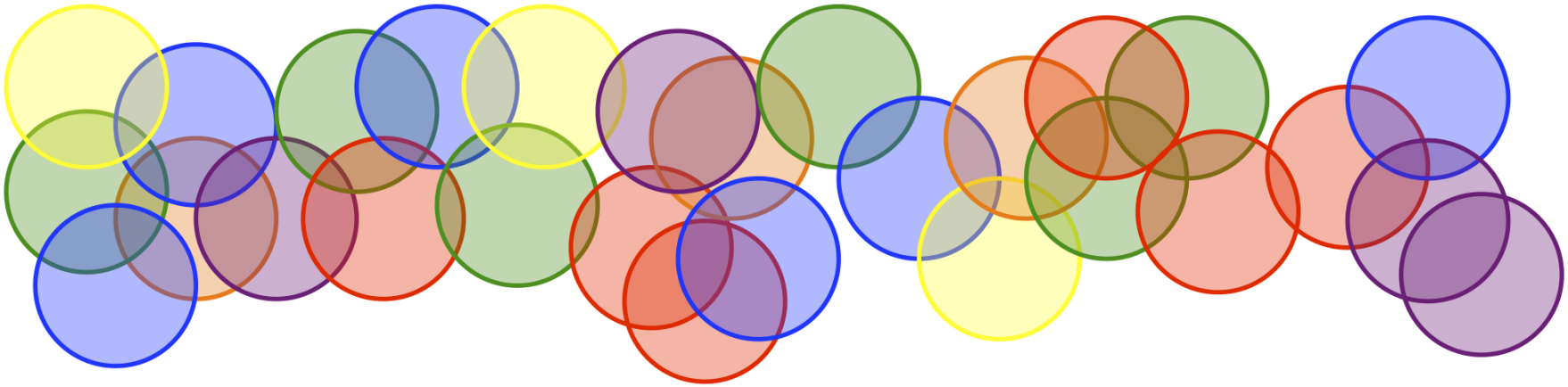

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  - **Colors**: Each sensor (camera) can "see" only one color.
  - **Sounds**: Each sensor (microphone) can only "hear" certain frequencies.
  - **Other Signatures**: Heat, Velocity, any property that has a linear structure.

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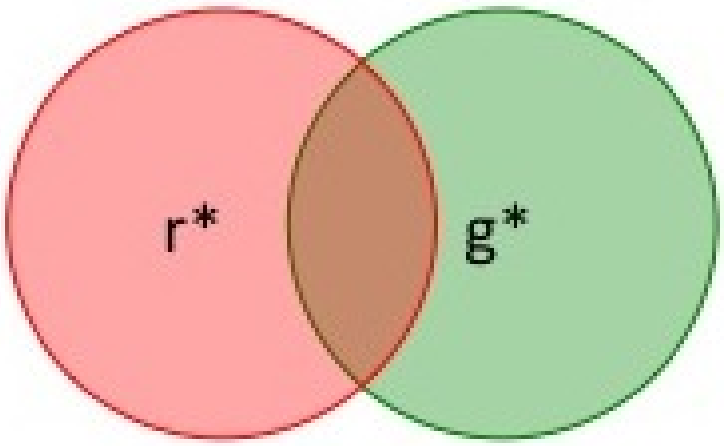


$$r^* \rightarrow < r^*, g^* > \leftarrow g^*$$

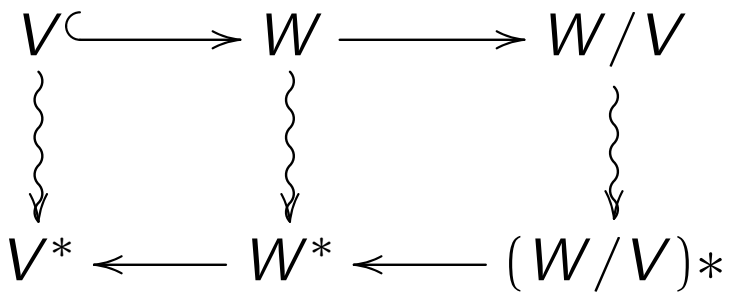
$$k^2/\operatorname{ann}(r^*) \leftarrow k^2 \rightarrow k^2/\operatorname{ann}(g^*)$$

- Thus we have to use **sheaves and cosheaves**!

What is a Sensor Really?

- What evades? Given $V \subseteq W$

$$\mathrm{ann}_W(V) = \{\varphi \in W^* | \varphi(v) = 0 \, \forall v \in V\} \cong (W/V)^*$$



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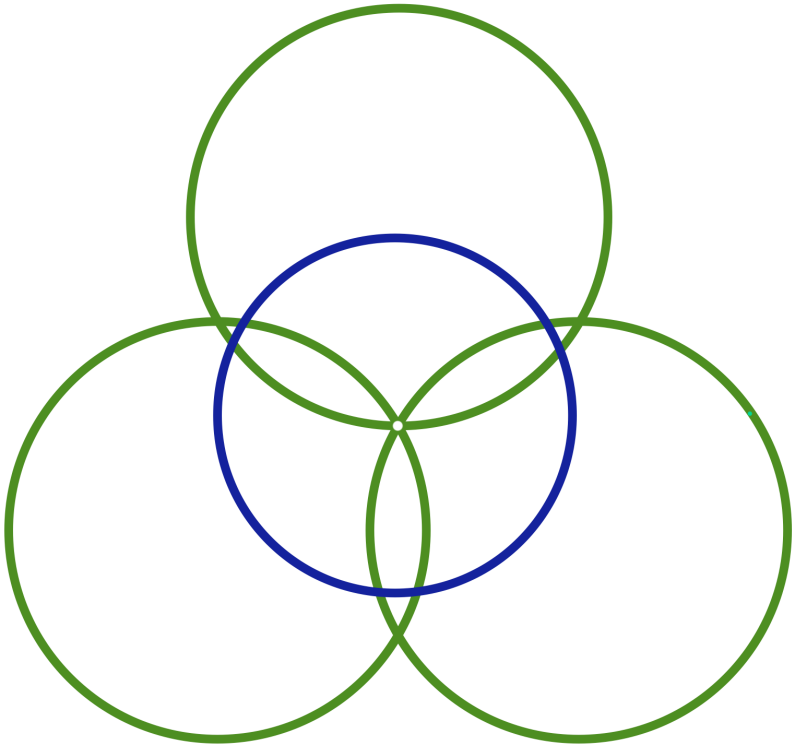
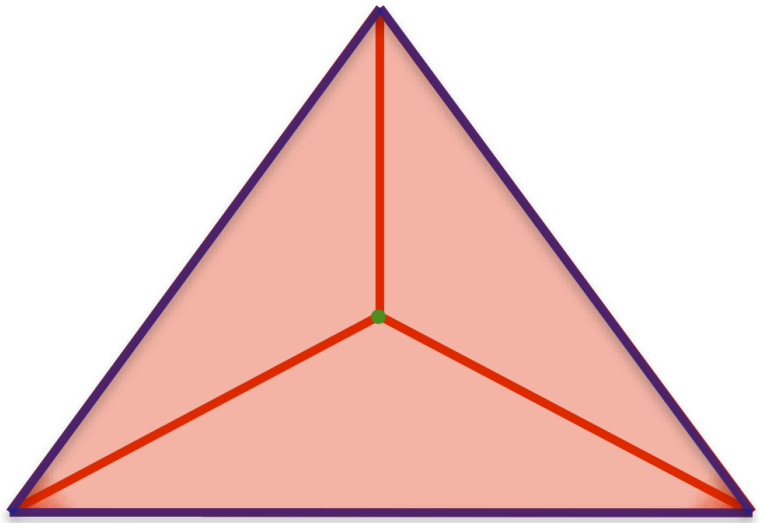
- And an **evasion cosheaf**

$$\widehat{E}(e_{vw}) = \mathrm{ann}(F(e_{vw})) \to \mathrm{ann}(F(v))$$

$$0 \to F \to \mathbf{k}_X^n \to \mathrm{cok} \to 0 \qquad \star(\mathrm{cok}) \cong \widehat{E}$$

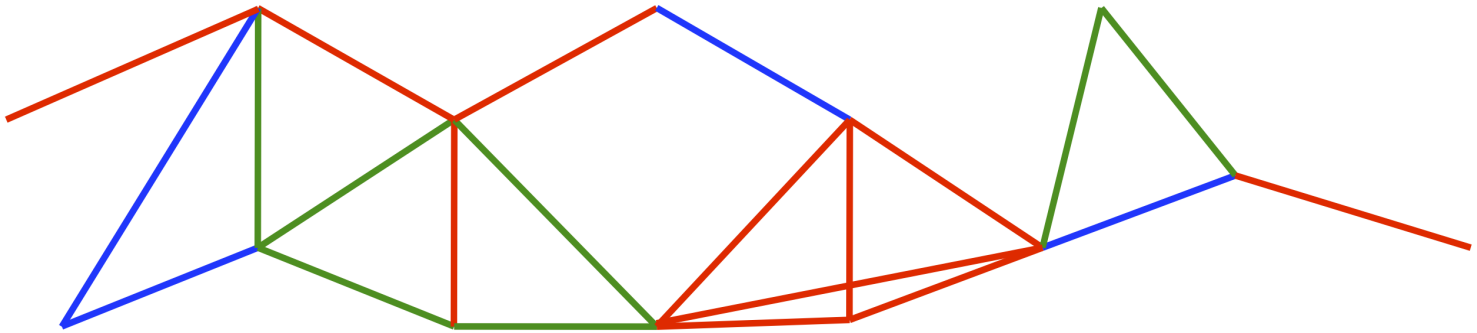
Example: Cosheaf Homology





- At center of blue sensor green and red can evade.
- At centers of green sensors and on edges red and blue can evade.
- No triple green intersection $\Rightarrow$ blue evasion set has non-trivial H_1.
- $\star$ Cellular cosheaf homology reveals $H_0(X;E) = \mathbb{R}^3$ and $H_1(X;E) = \mathbb{R}$. Coincidence?

No Coincidences



Theorem (Splitting Theorem for Evasion Cosheaves)

Suppose all the sensing capabilities come from a fixed orthonormal basis of k^{n*}. Let K_{e_i} be the subcomplex where an intruder with property vector e_i can evade detection. We then have

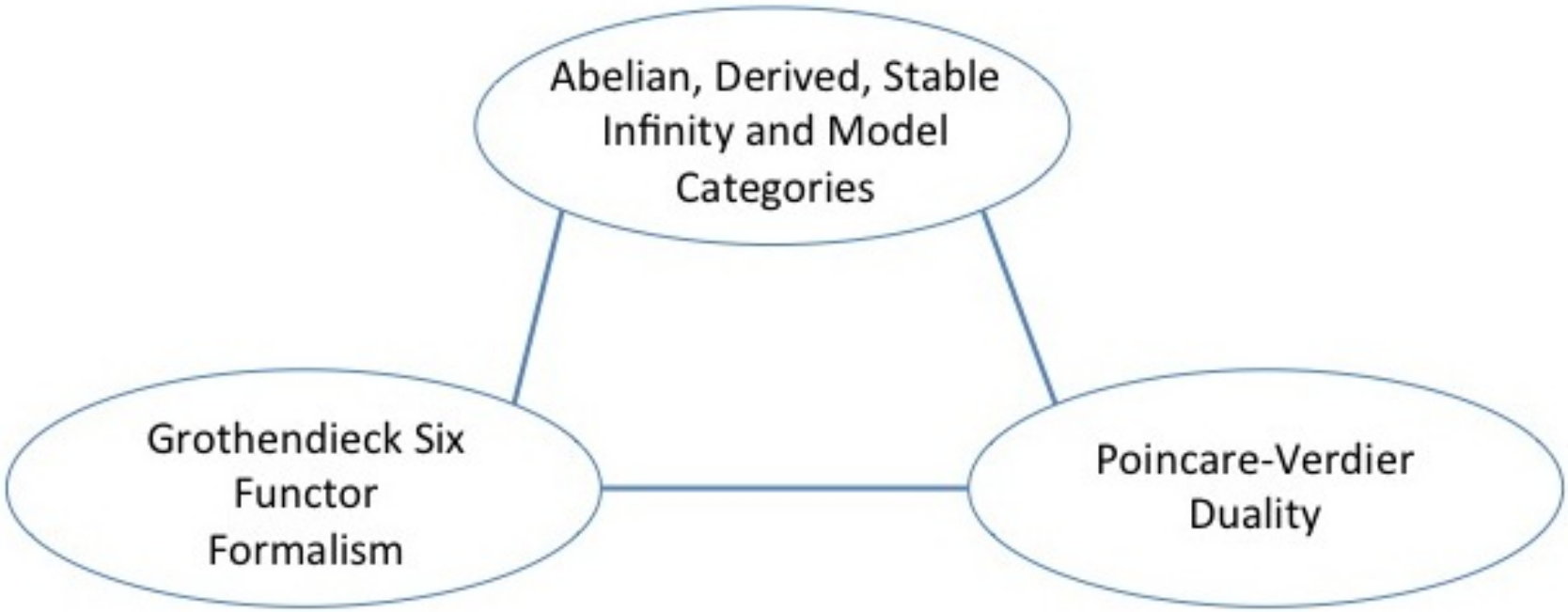
$$\begin{aligned} \widehat{E} &\cong \widehat{K}_{e_1} \oplus \cdots \oplus \widehat{K}_{e_n} \\ H_i(X; \widehat{E}) &\cong H_i(K_{e_1}; \mathbb{R}) \oplus \cdots \oplus H_i(K_{e_n}; \mathbb{R}) \end{aligned}$$

Theorem (Topological Forcing for Multi-modal sensing)

Given a sensing sheaf of vector spaces $\iota : F \to G = \tilde{k}_X^{n*}$ we obtain a long exact sequence of sheaf cohomology groups

$$0 \longrightarrow H^0(X;F) \longrightarrow H^0(X;k)^{\oplus n} \longrightarrow H^0(X;cok(\iota)) \xrightarrow{\delta^0}$$
$$H^1(X;F) \longrightarrow \cdots \longrightarrow H^k(X;cok(\iota)) \xrightarrow{\delta^k}$$
$$H^{k+1}(X;F) \longrightarrow H^{k+1}(X;k)^{\oplus n} \longrightarrow H^{n+1}(X;cok(\iota)) \xrightarrow{\delta^{k+1}}$$
$$\longrightarrow \cdots$$

where $H^k(X;cok(\iota)) \cong H_k(X;\widehat{E})^*$.



- Apply this package of tools to: Persistent Homology, Network Coding, Control Theory...
- Get **(co)homological invariants** and can ask **new questions** like: *What is the associated (Verdier dual) sheaf to persistence?*

Acknowledgements

- Primary thanks to Rob Ghrist, David Lipsky, Sanjeevi Krishnan, Bob MacPherson
- Interesting related work by Henry Adams, Gunnar Carlsson, Amit Patel, Michael Robinson, Radmila Sazdanovic and many others.

Special Thanks to the Organizers!