The Adaptive Topology of a Digital Image

ATMCS' 2012

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IST Austria, Duke & Geomagic

Part I The Difficulty

Part II The Motivation

Part III The Solution

I.1 Persistence Diagram

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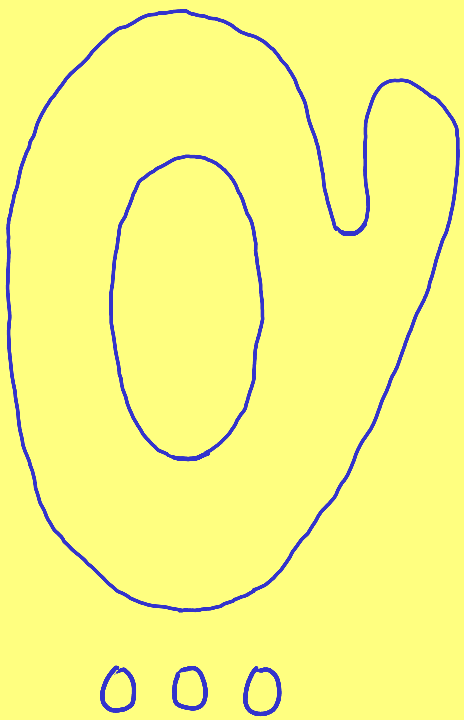

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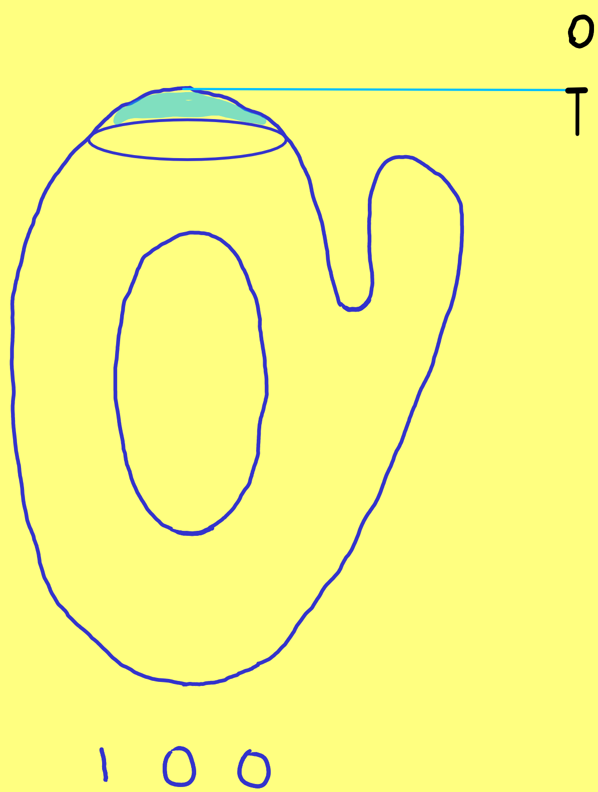

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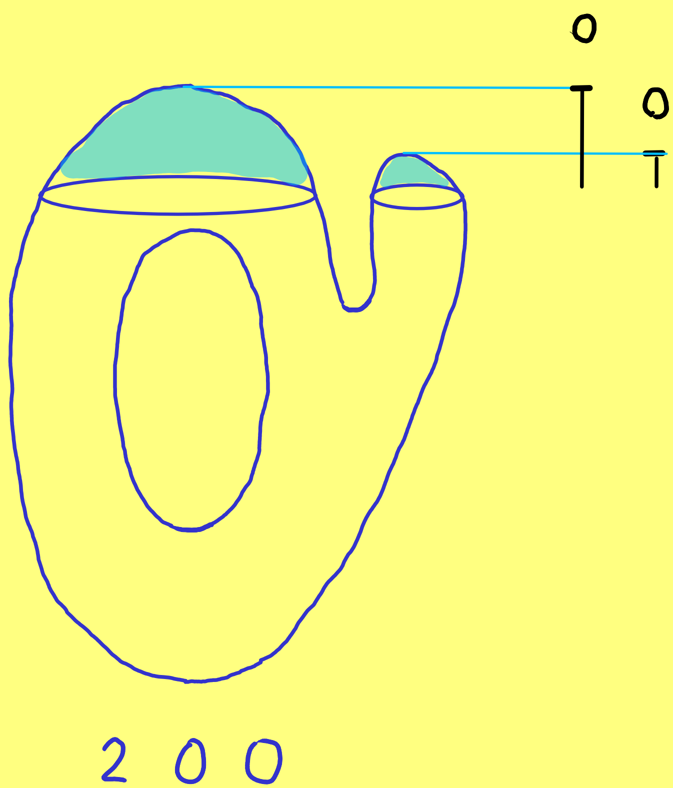

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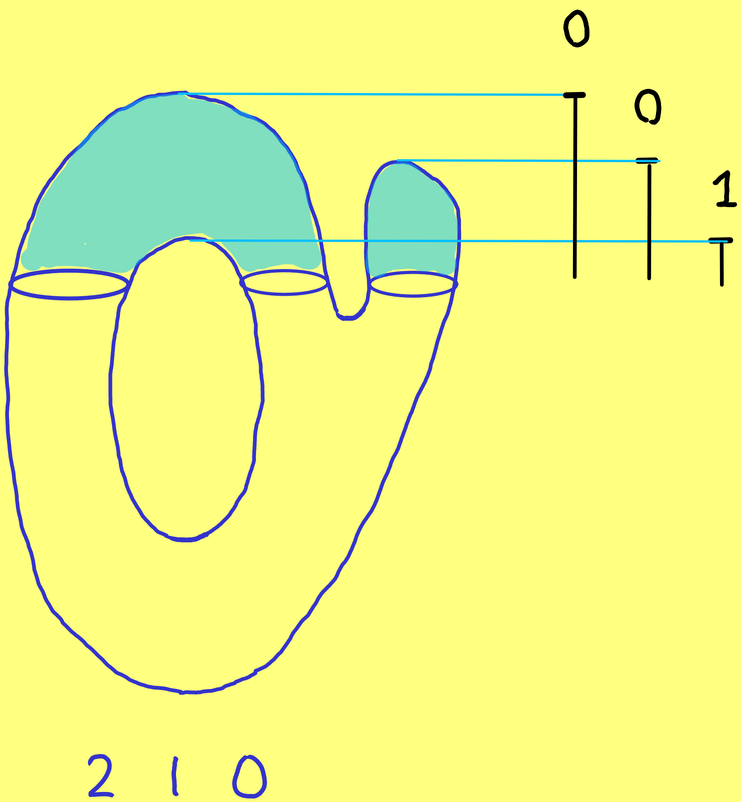

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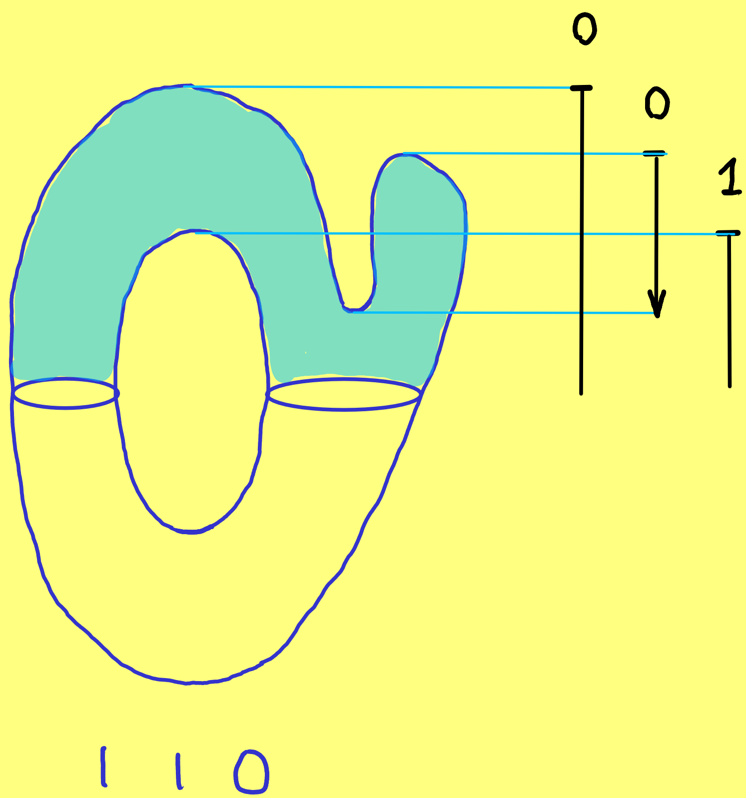

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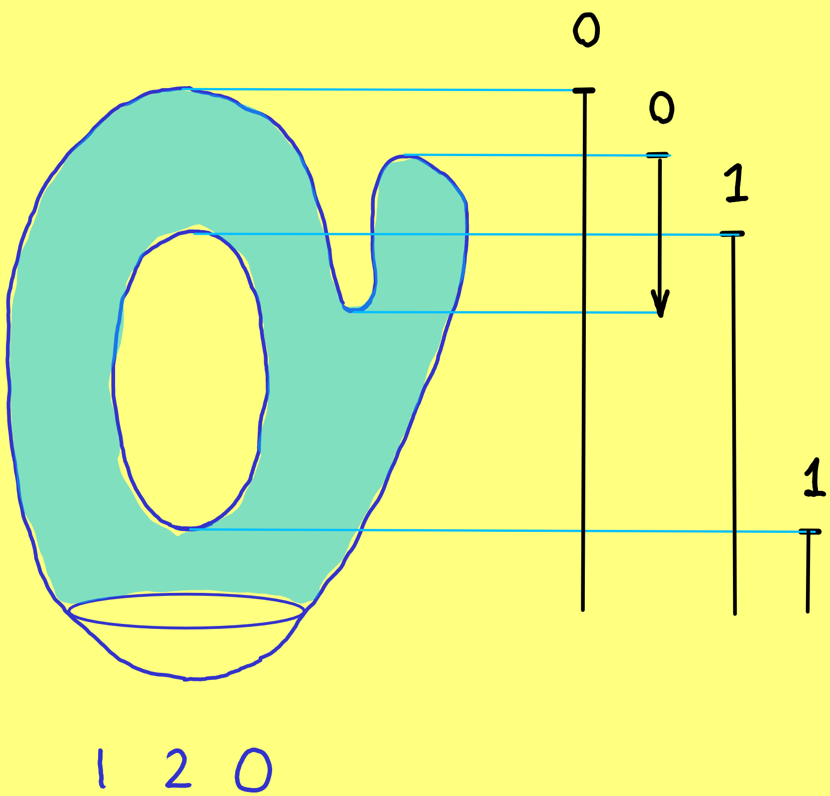

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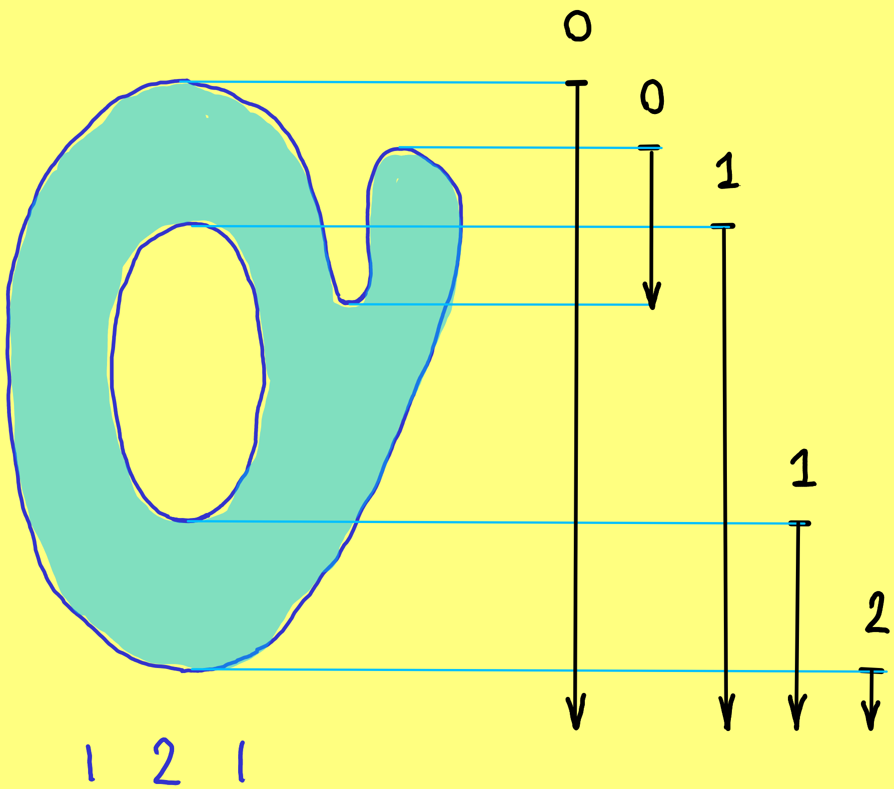

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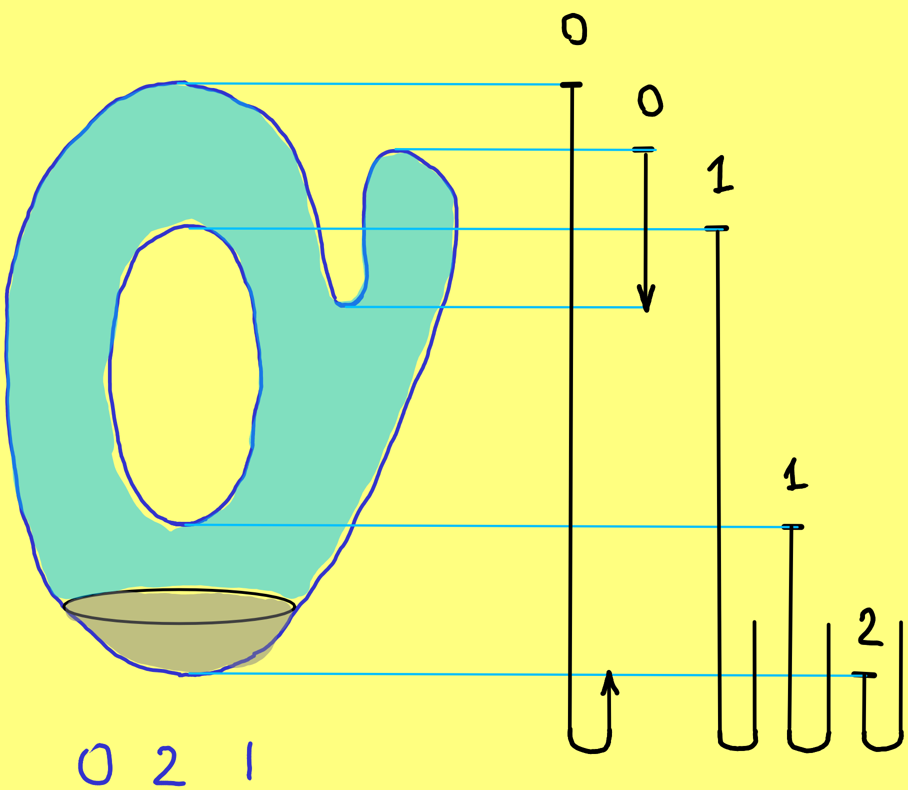

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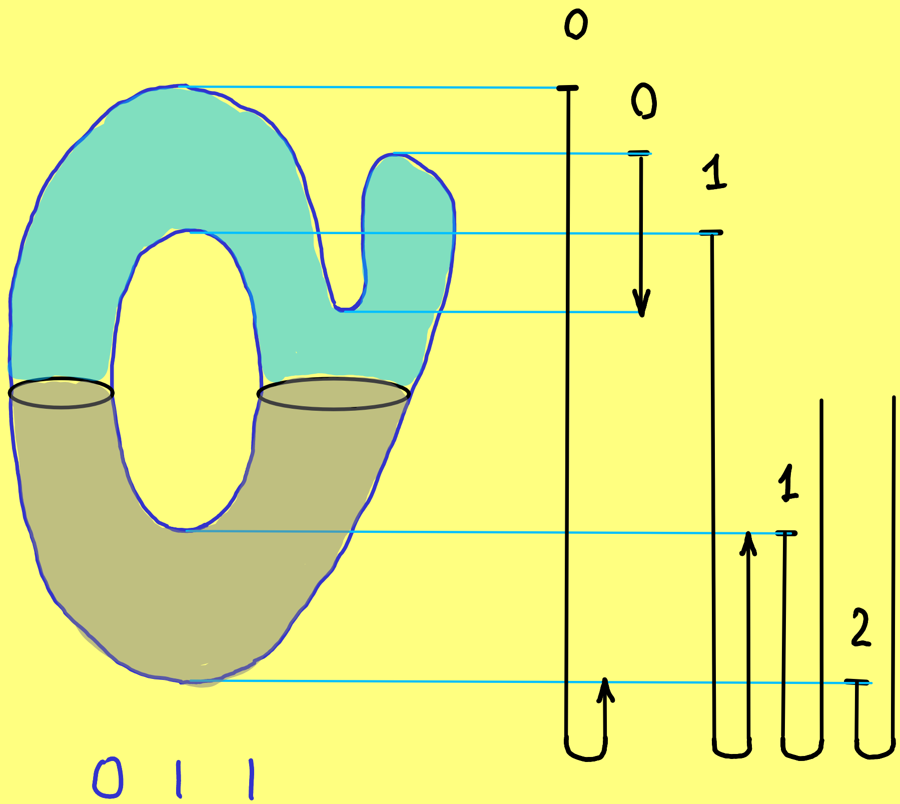

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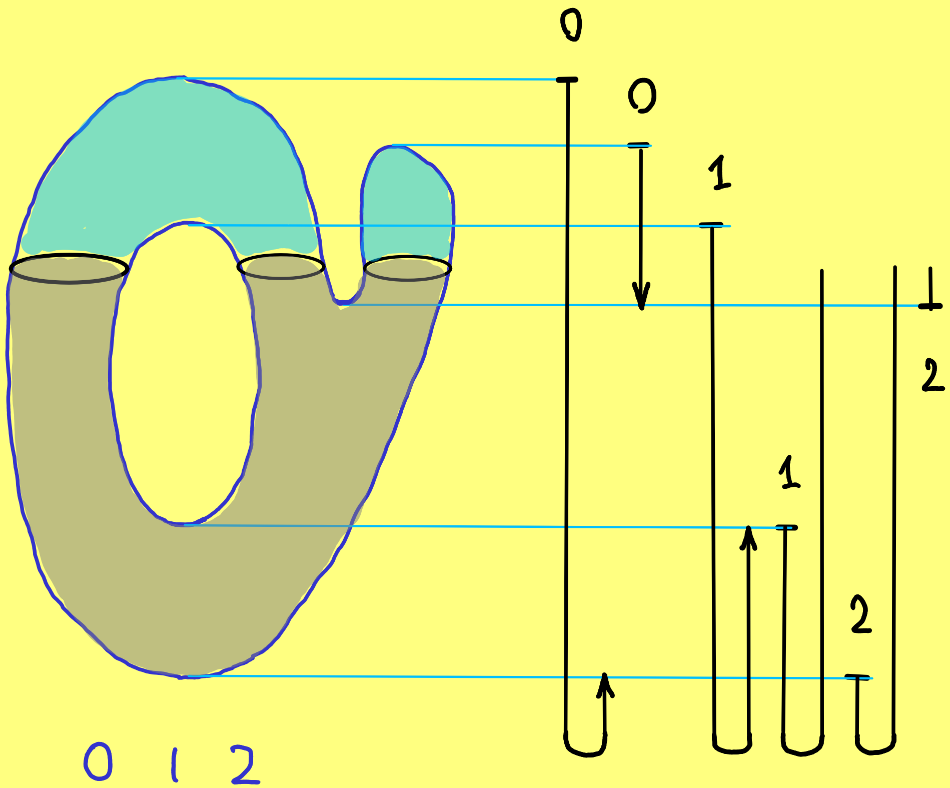

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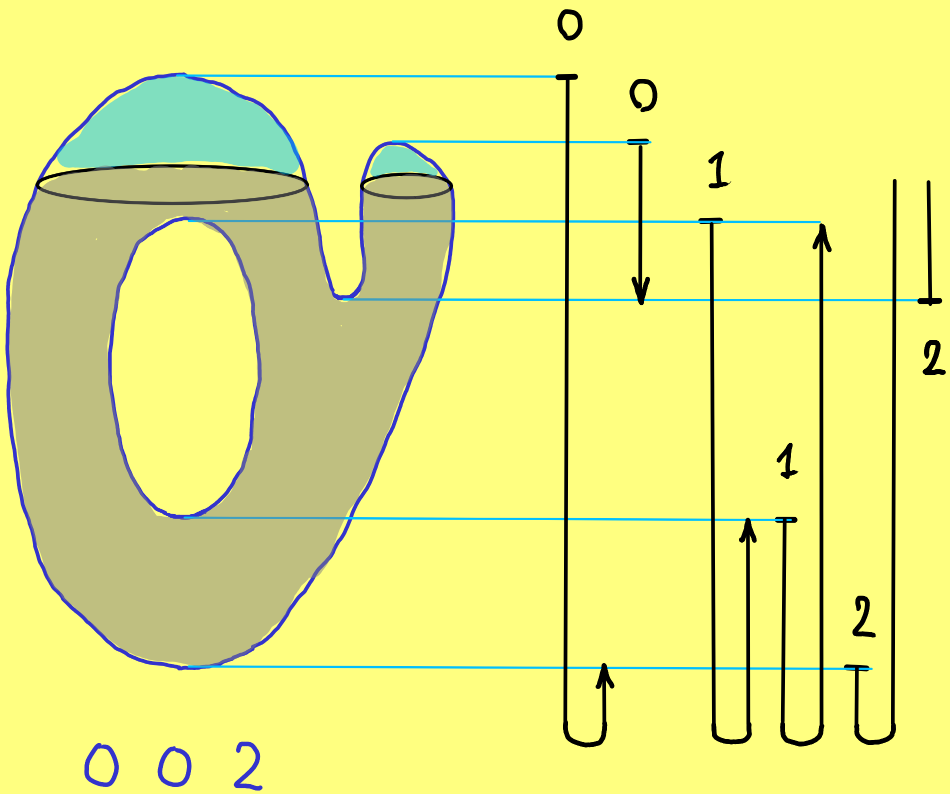

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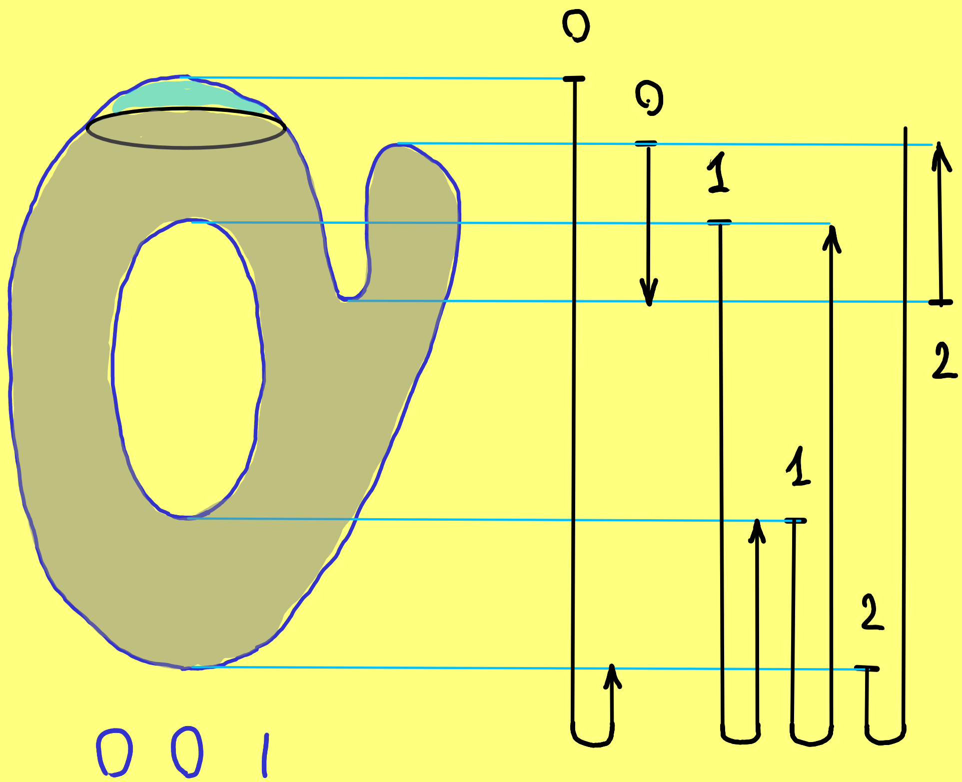

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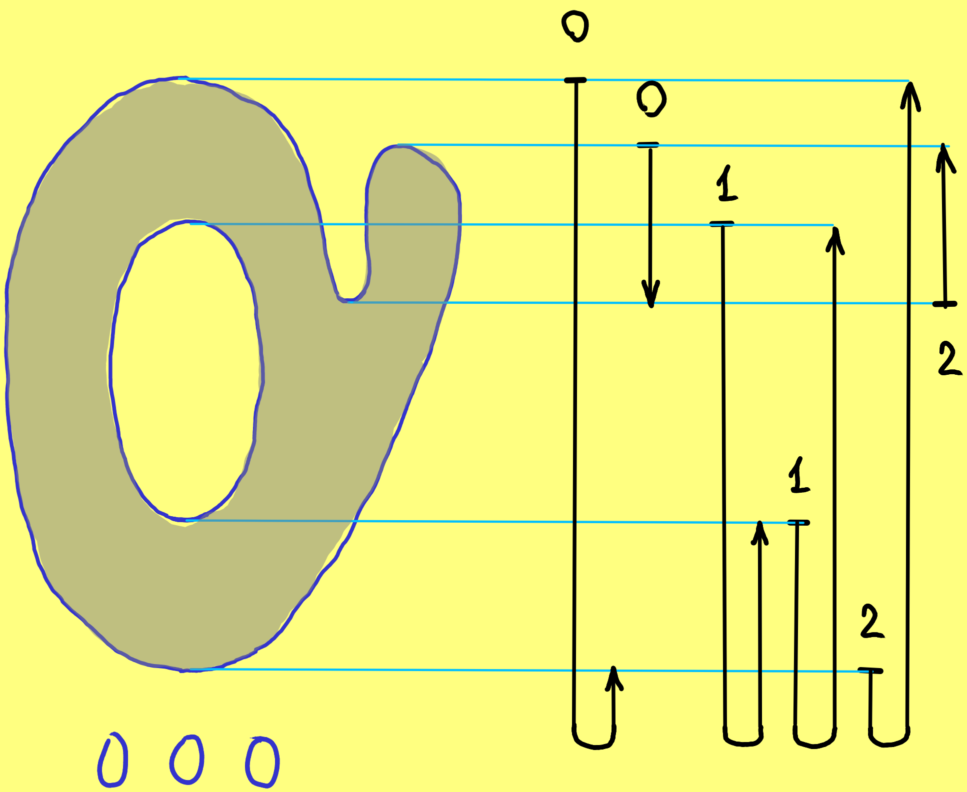

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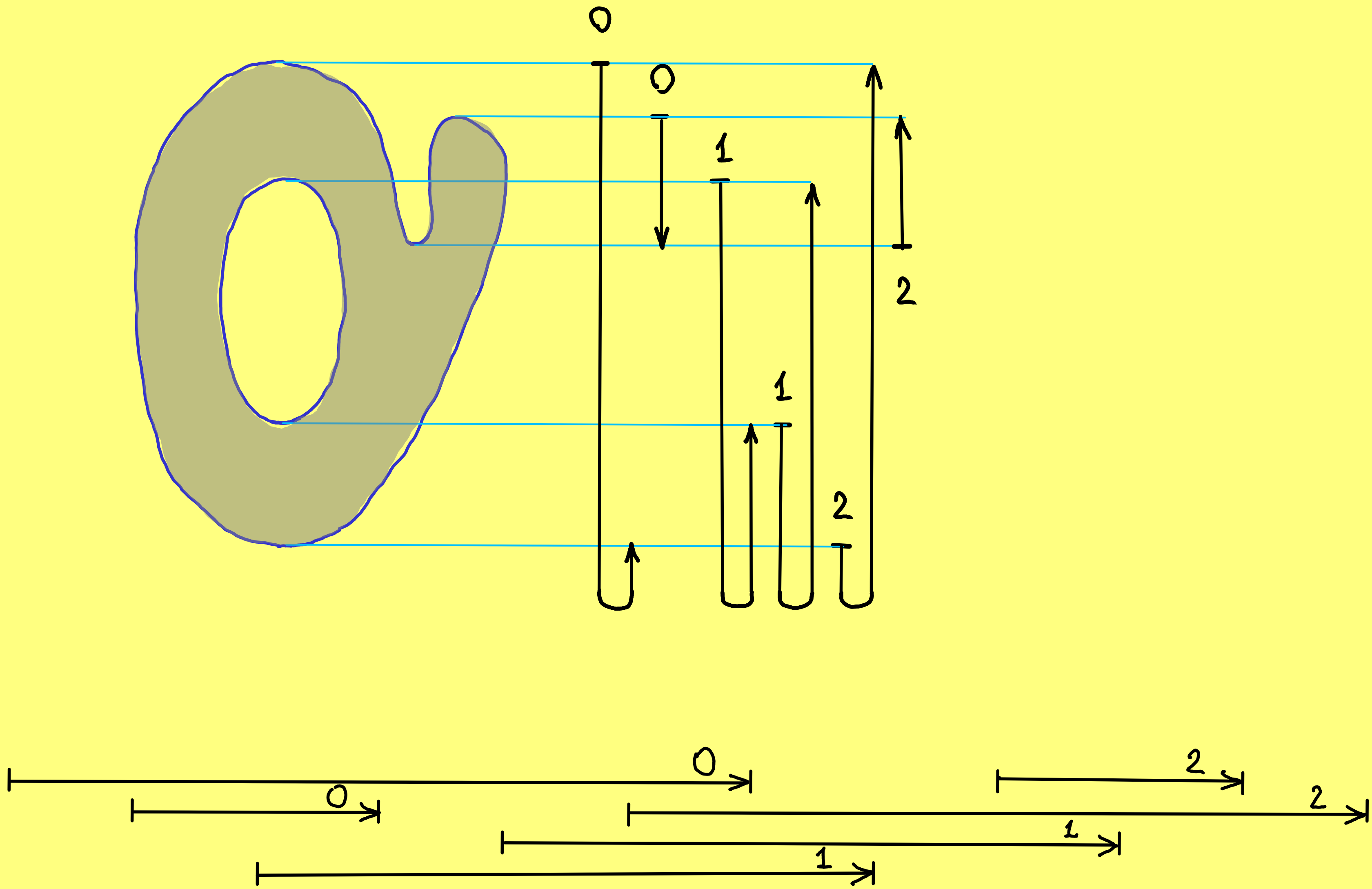

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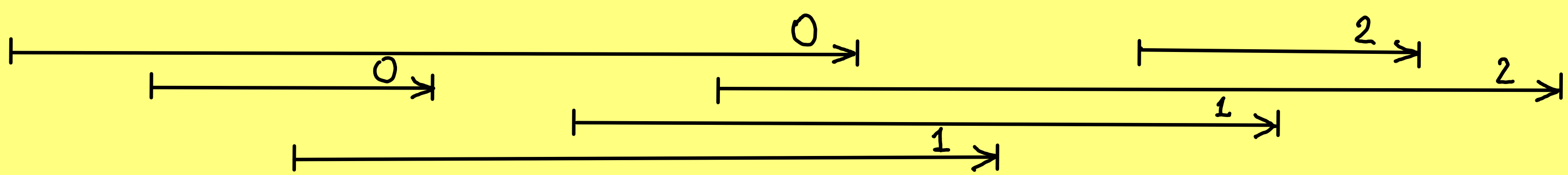

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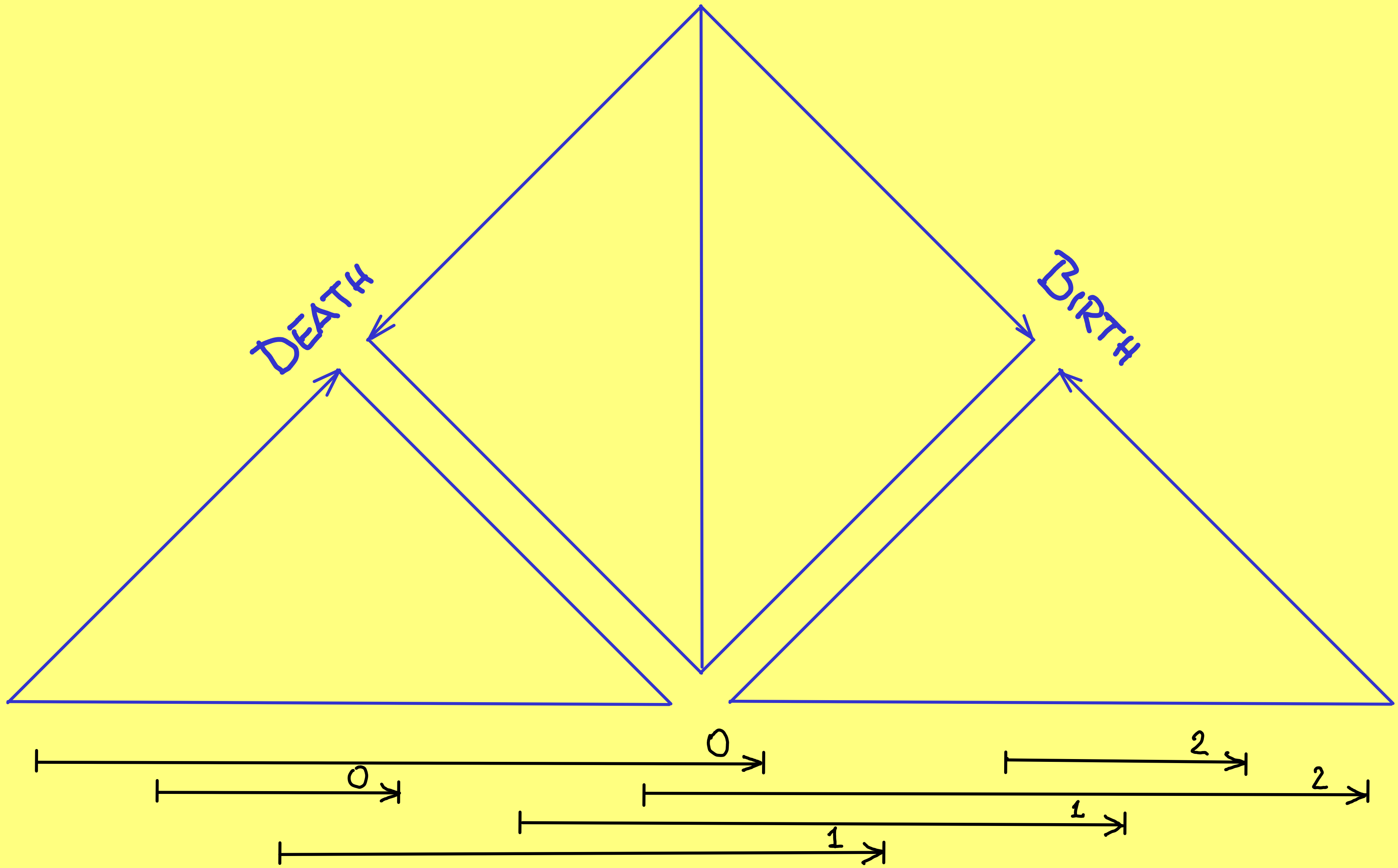

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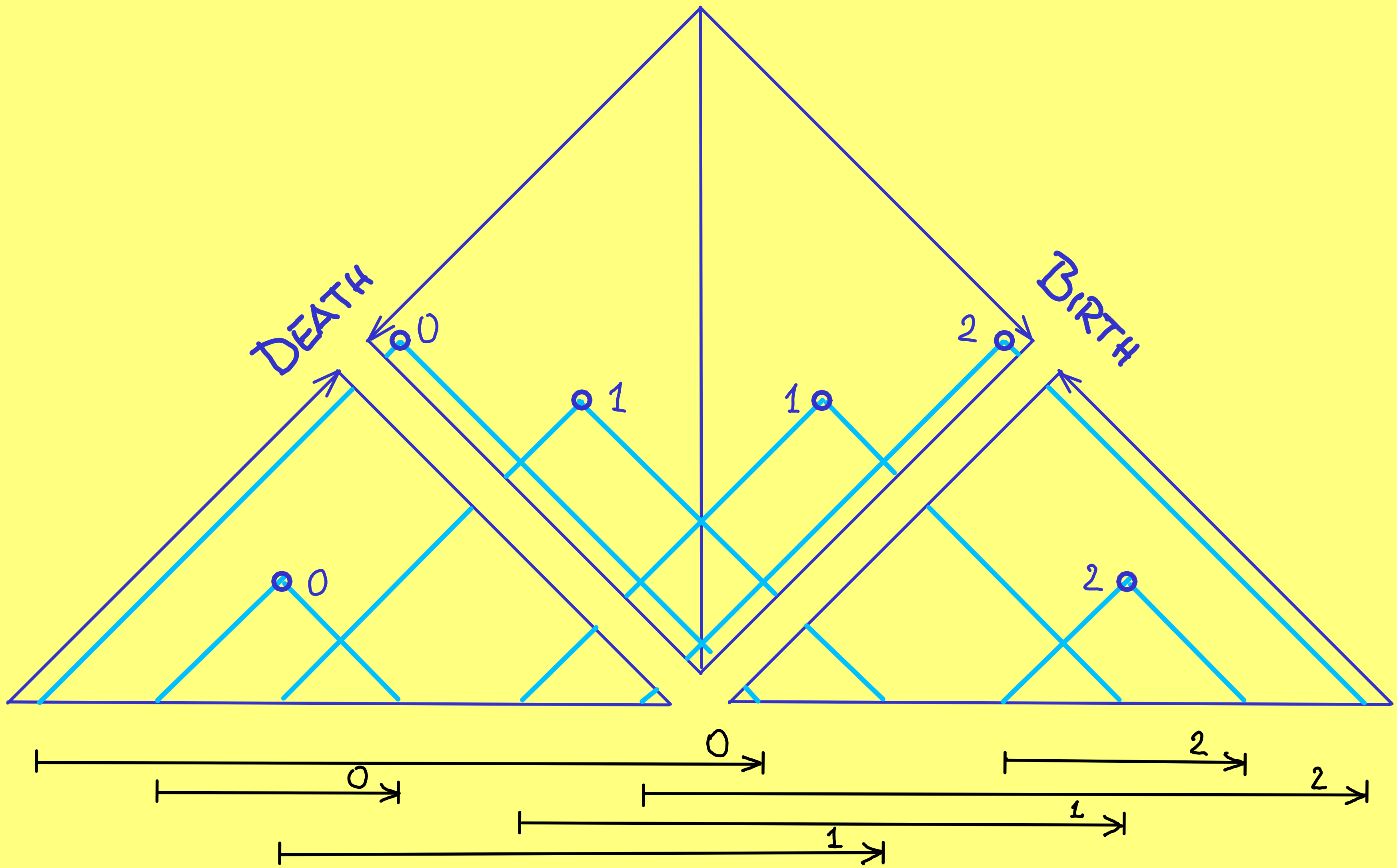

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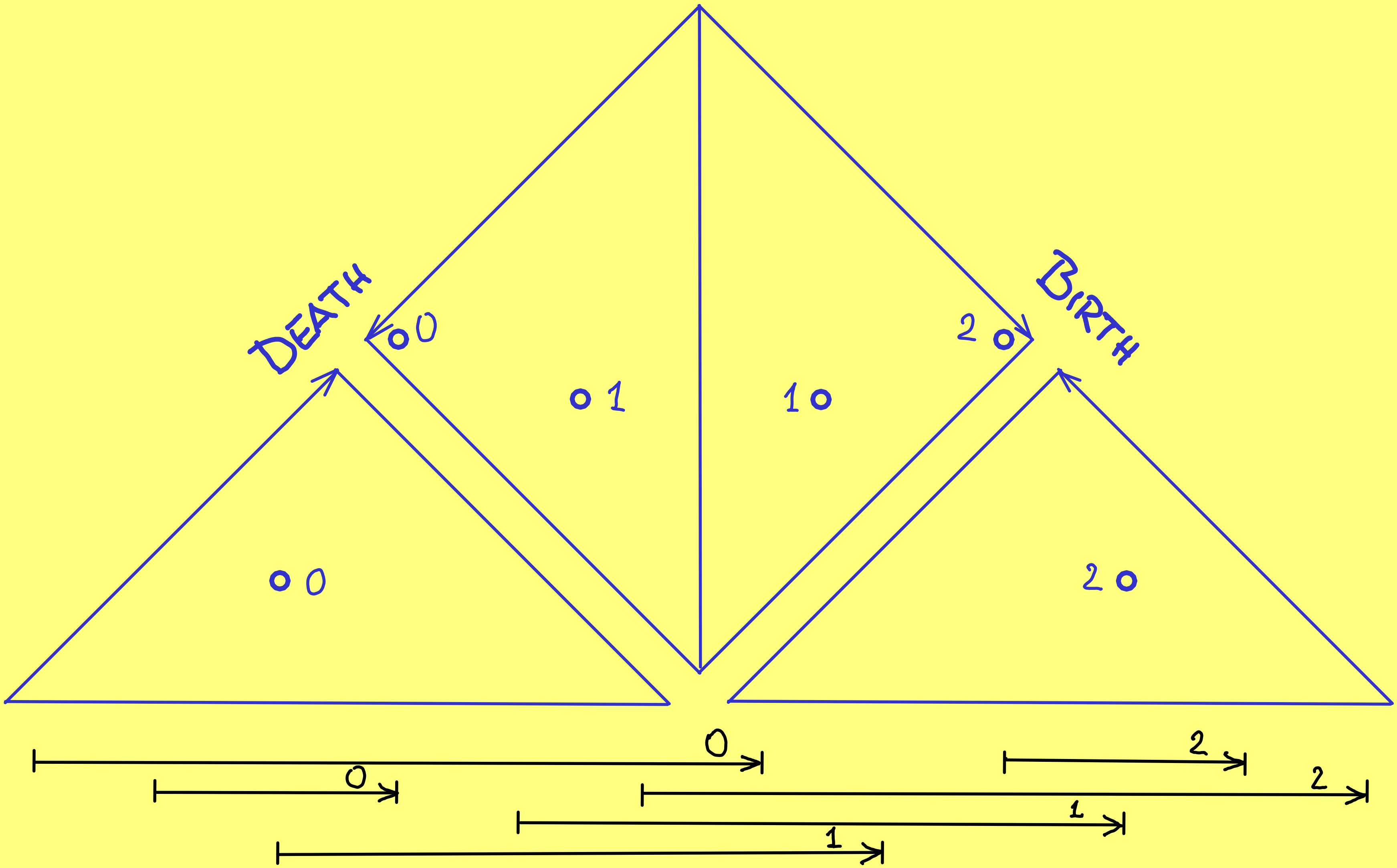
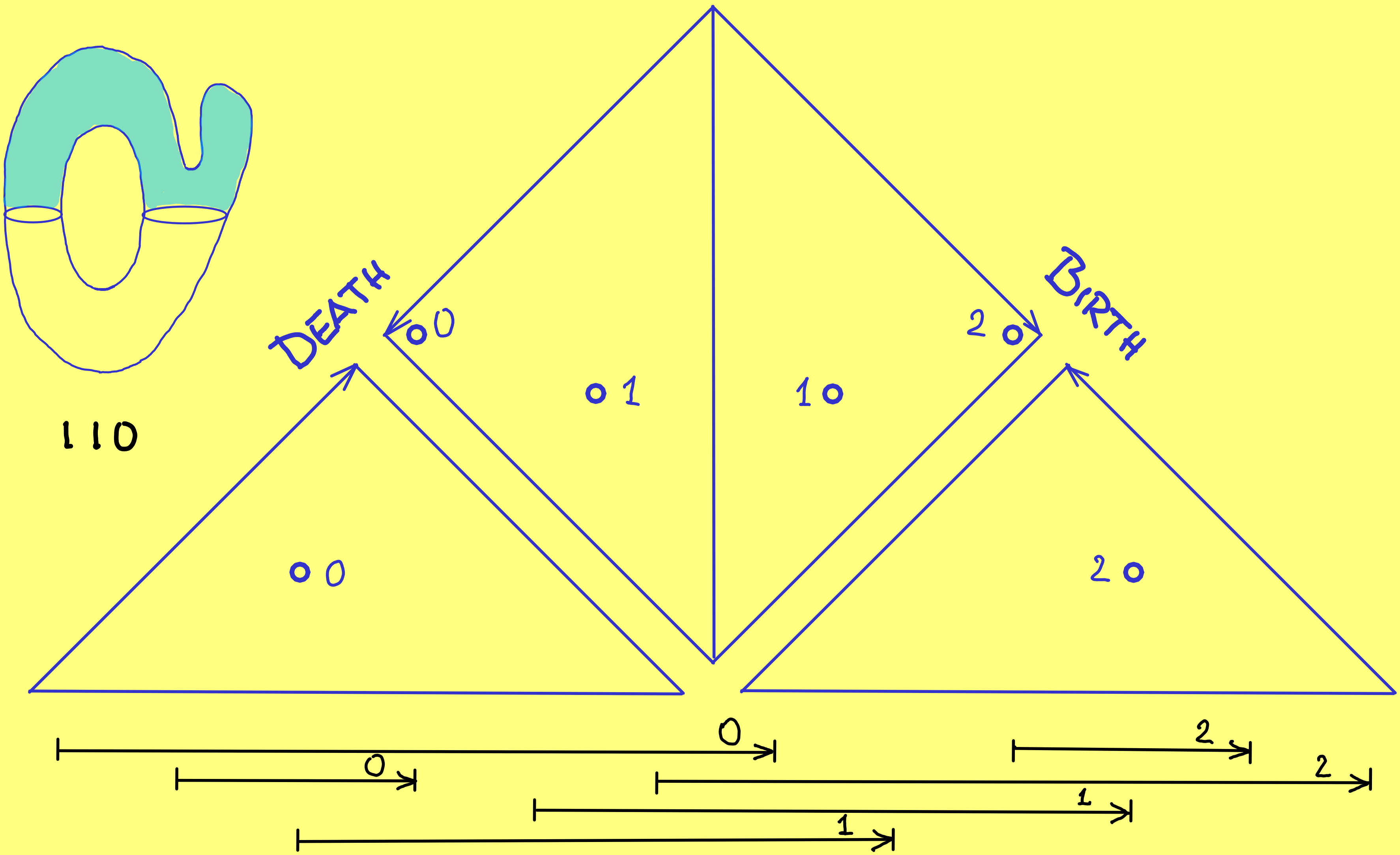


I.2 Retrieval



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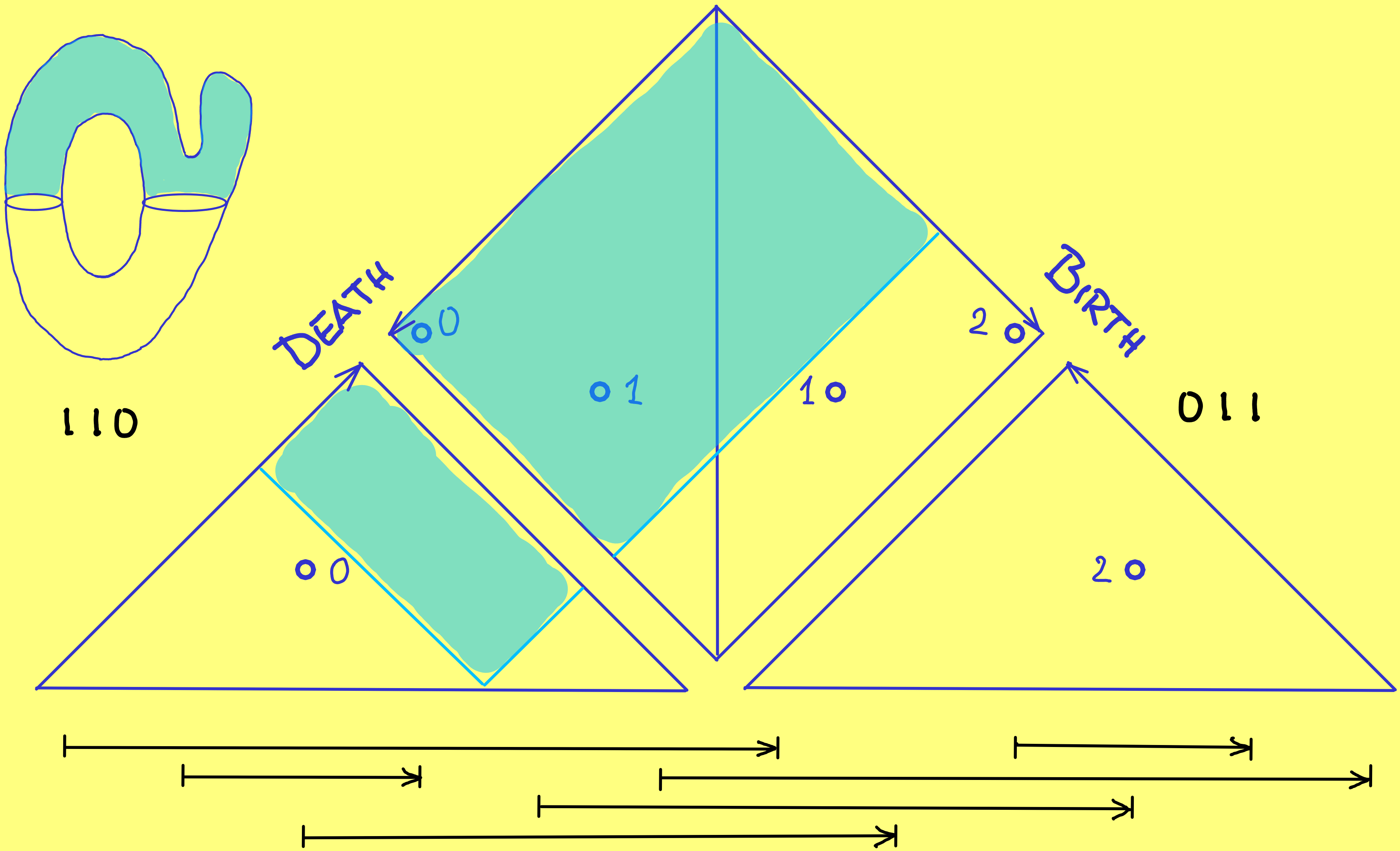

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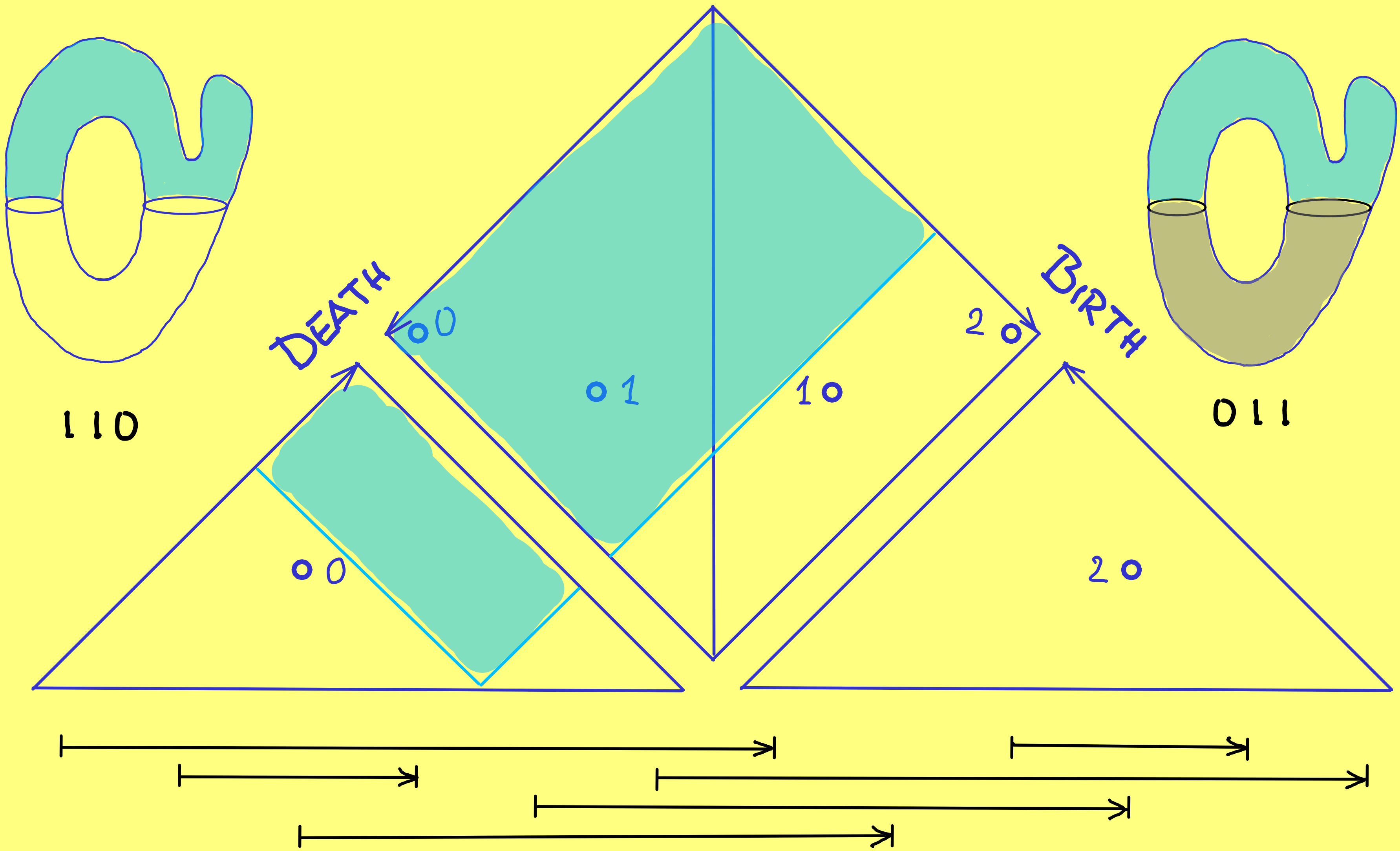

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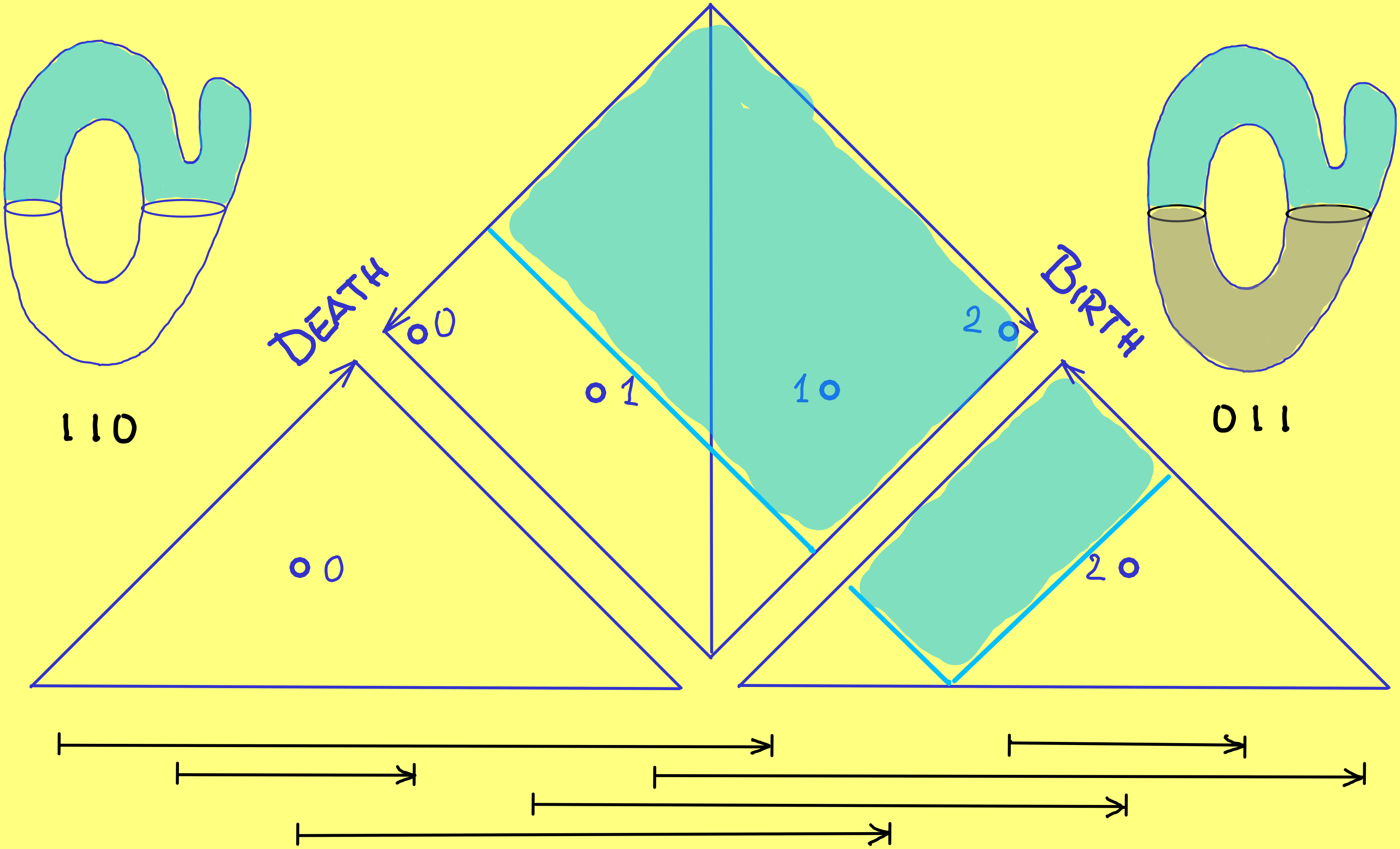

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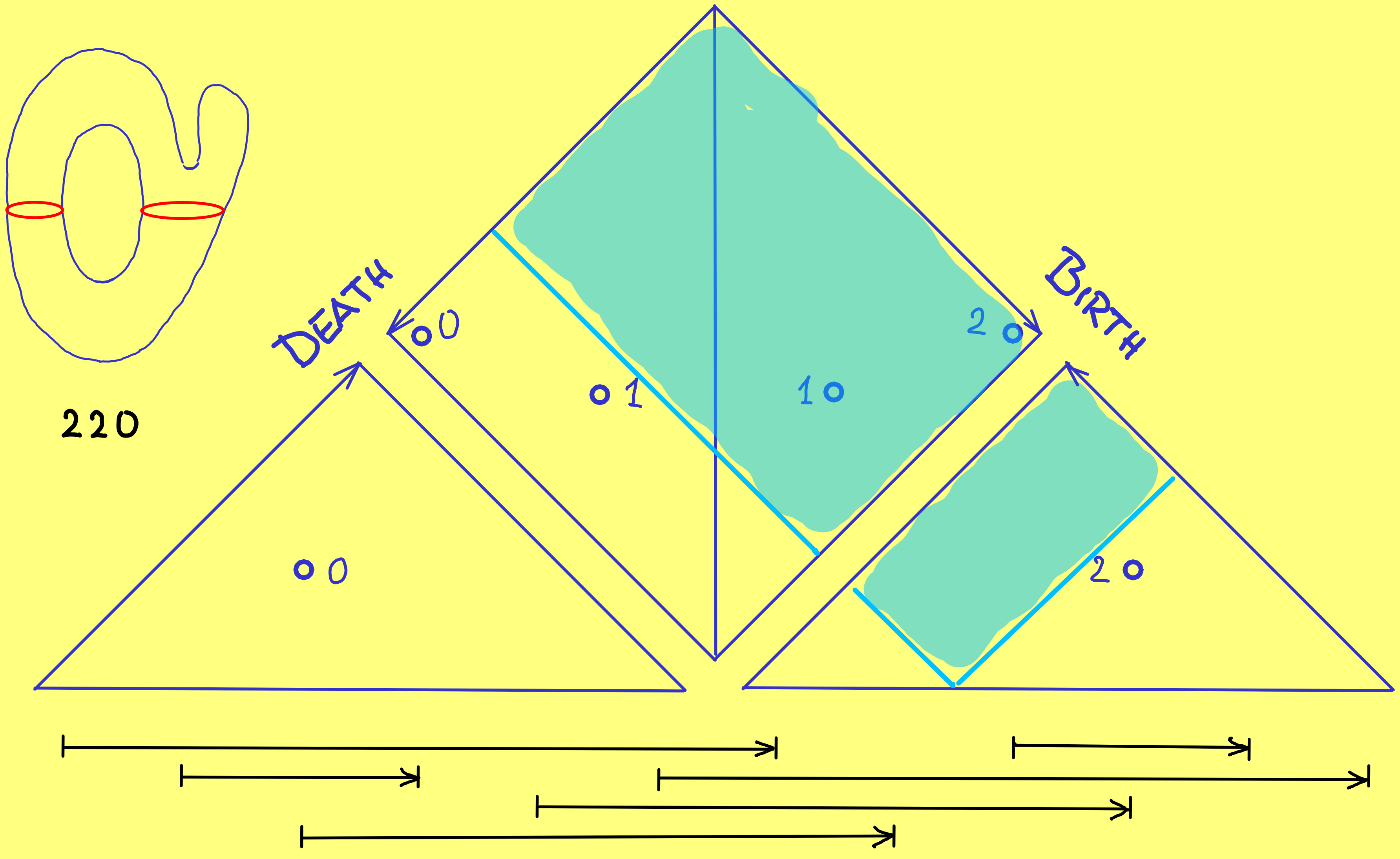

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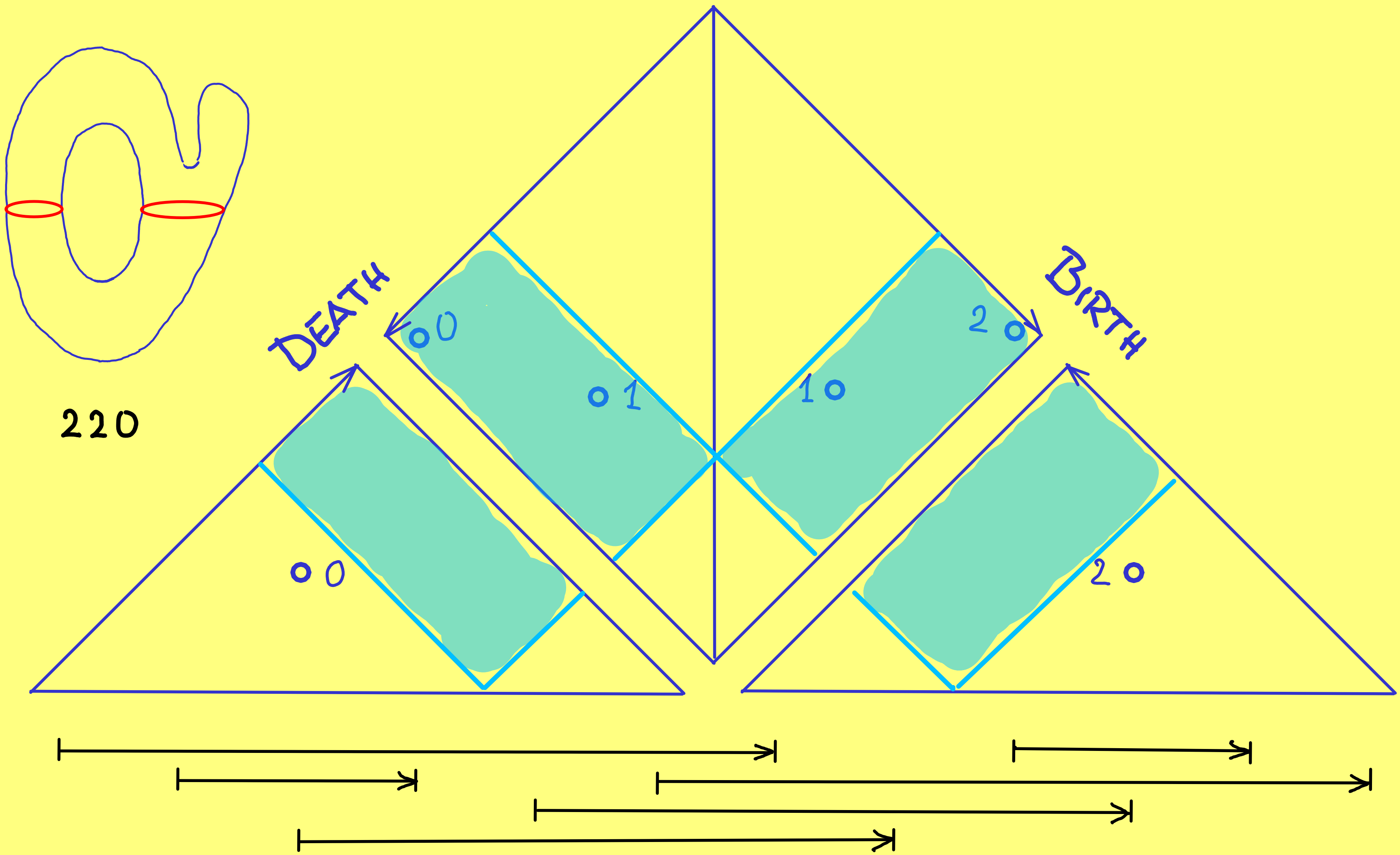

I.3 SYMMETRIES

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(i) $\mathrm{Dgm}(f) = \mathrm{Dgm}^T(f)$ if $\mathbb{X}$ is a manifold

[Lefschetz Duality]

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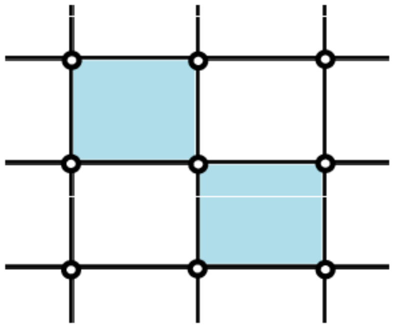
[Lefschetz Duality]

(ii) $\mathrm{Dgm}(f|\partial \mathbb{X}) = \mathrm{Dgm}(f|\mathbb{X}) \sqcup \mathrm{Dgm}^T(f|\mathbb{X})$ if $\mathbb{X} \subseteq \mathbb{R}^n$ compact with $\partial\mathbb{X}$ an $(n-1)$-manifold and $f: \mathbb{R}^n \to \mathbb{R}$ has no critical points

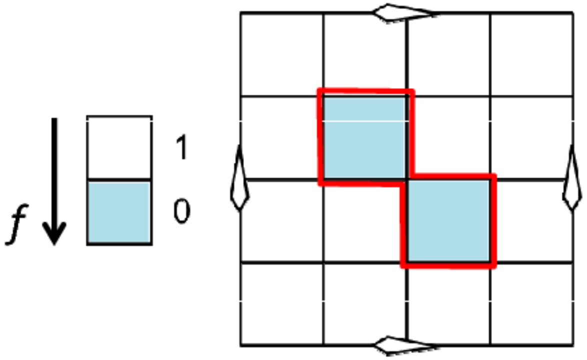
[Alexander Duality and Mayer-Vietoris]

Problem Statement

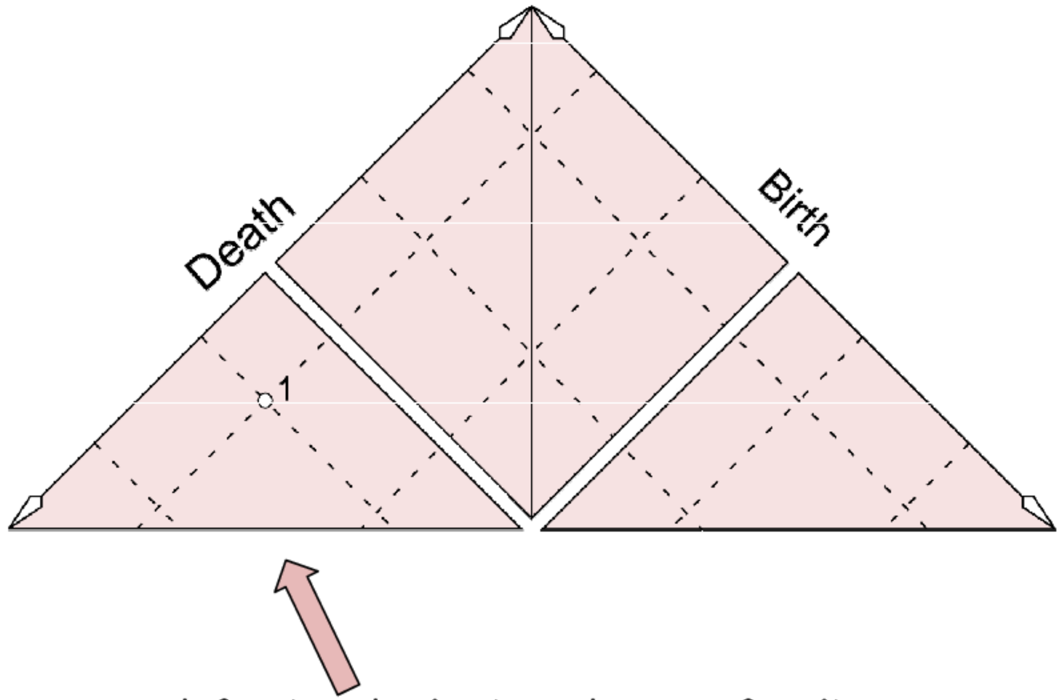
Images are often represented as cubical complexes.



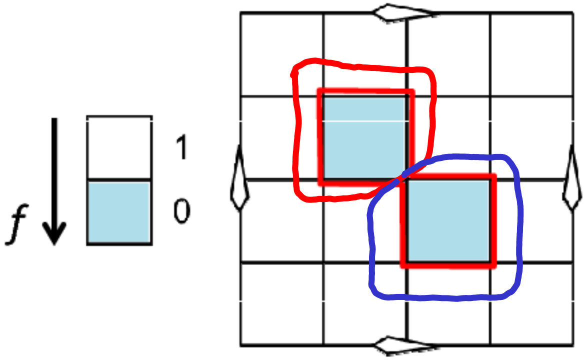
Computation of persistence diagrams for images represented as cubical complexes fails to preserve fundamental topological symmetries, such as Alexander duality.



A cycle is created at f=1 and destroyed (filled) at f=0



Lower left triangle depicts classes of ordinary homology: they are created and destroyed in the direction of function decrease, on the way down



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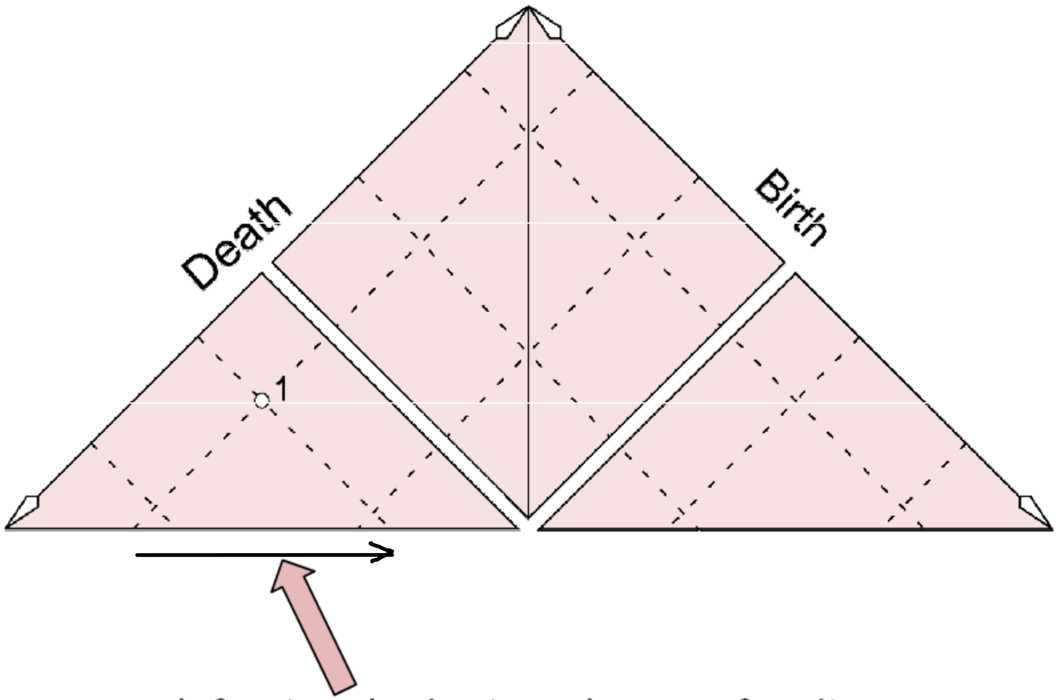

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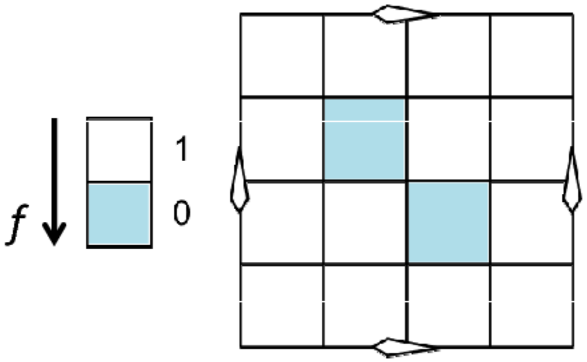
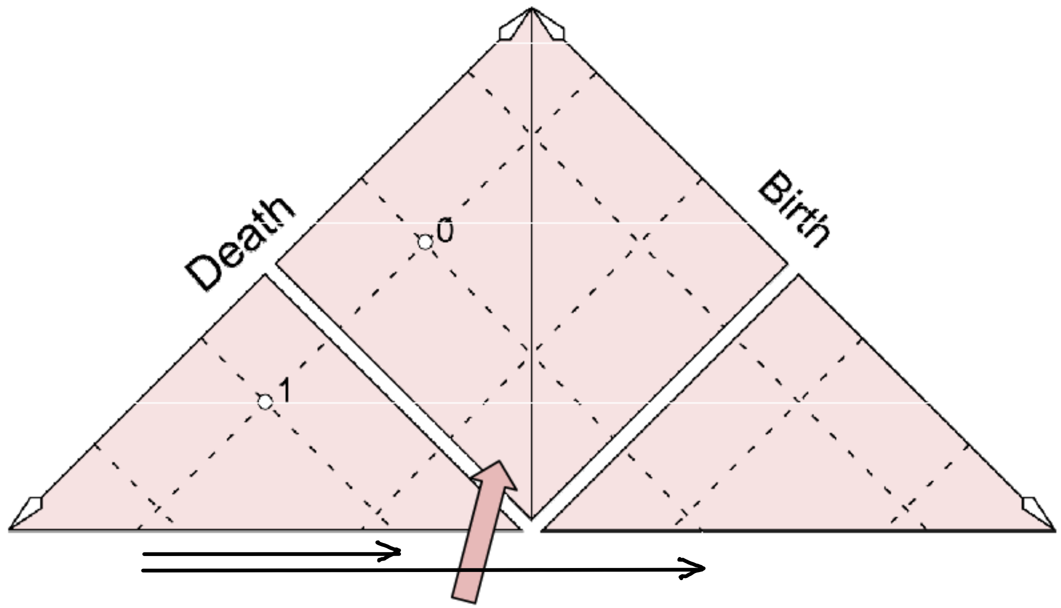
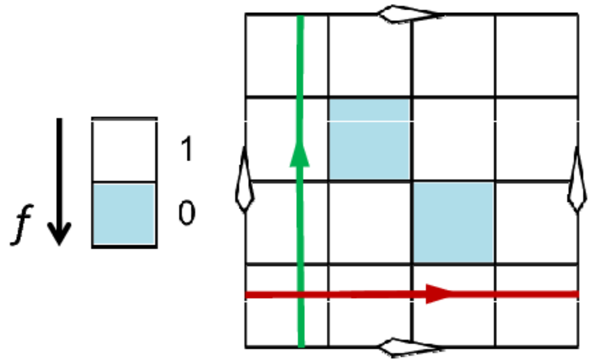


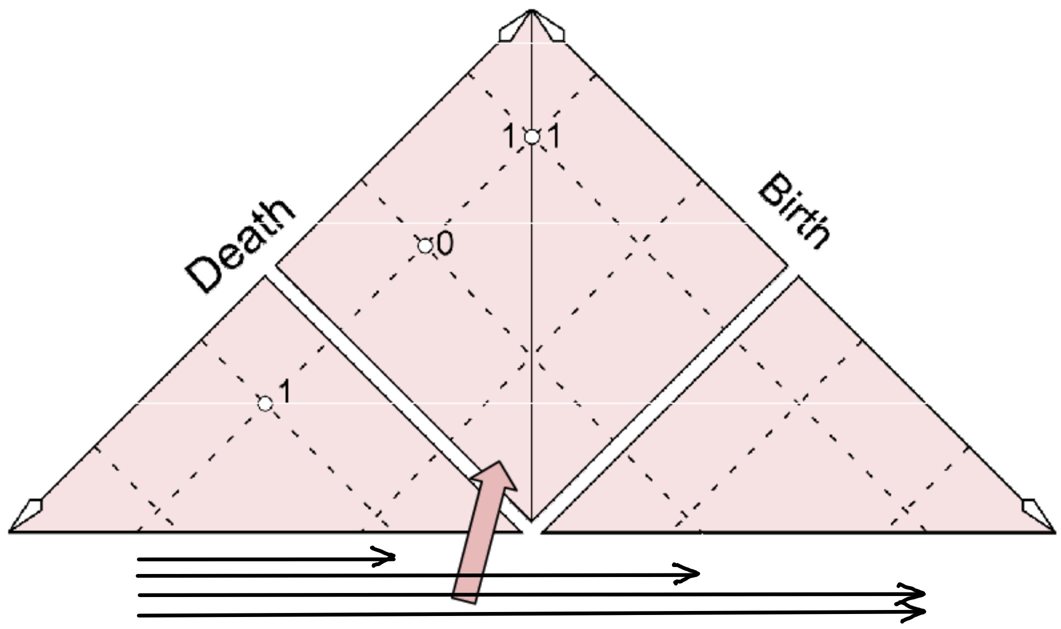
A component is created at *f*=1 on the way down. It is destroyed when we add blue cubes to the second covering, at *f*=0 on the way up.



Triangles in the middle depict essential homology classes: they are created on the way down and destroyed on the way up



Two cycles are created at *f*=1 on the way down. They are destroyed when we add all cubes to the second covering, at *f*=1 on the way up.



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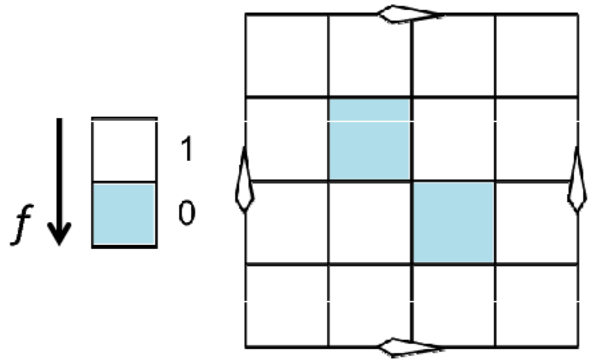
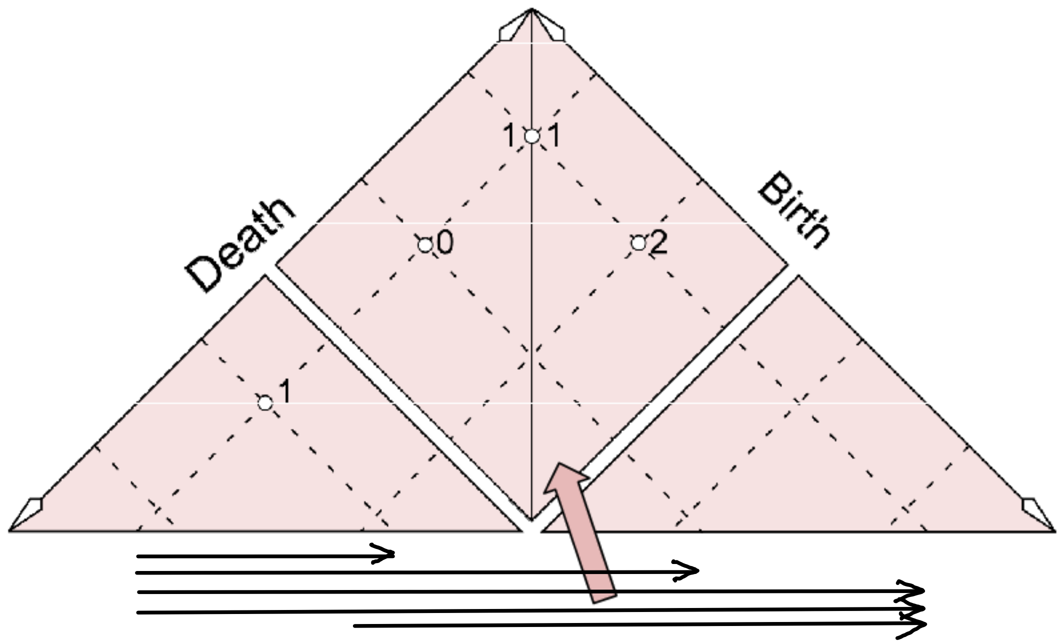


When we add blue cubes at *f*=1 on the way down, we create a void, a dim 2 homology class. It is destroyed when we add all cubes to the second covering, at *f*=0 on the way up.



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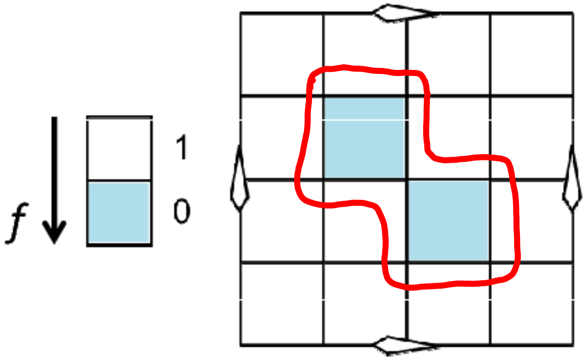
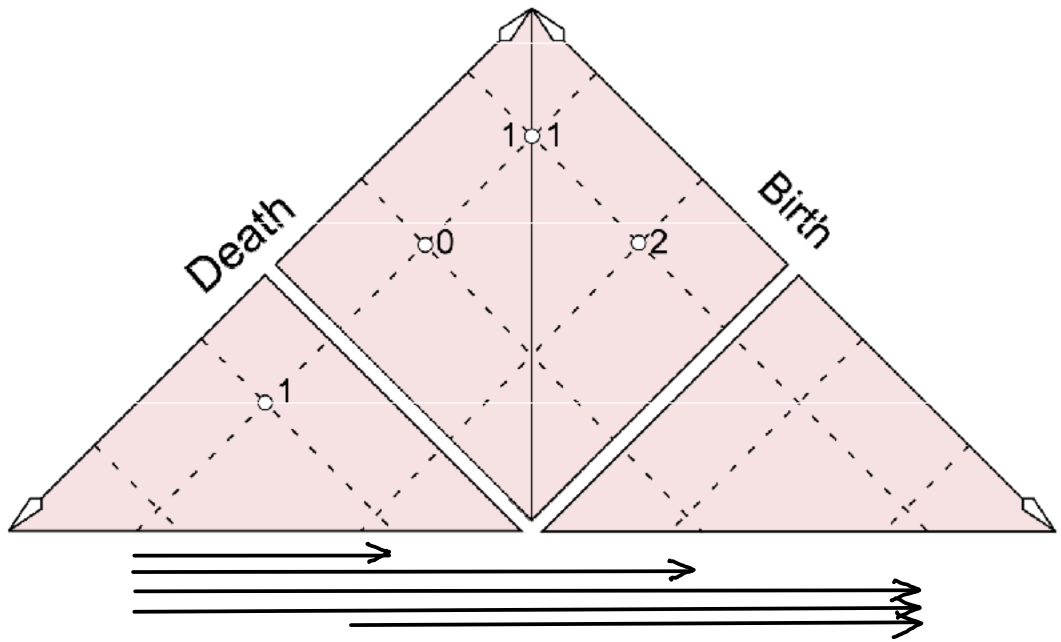
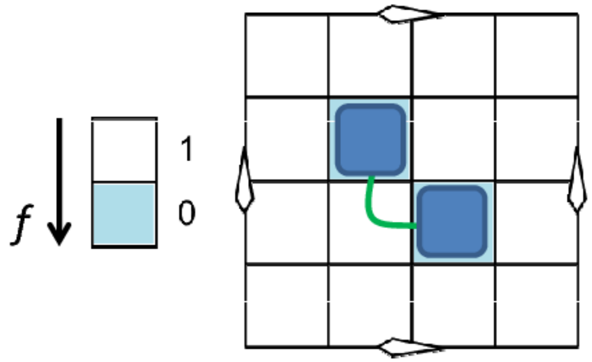


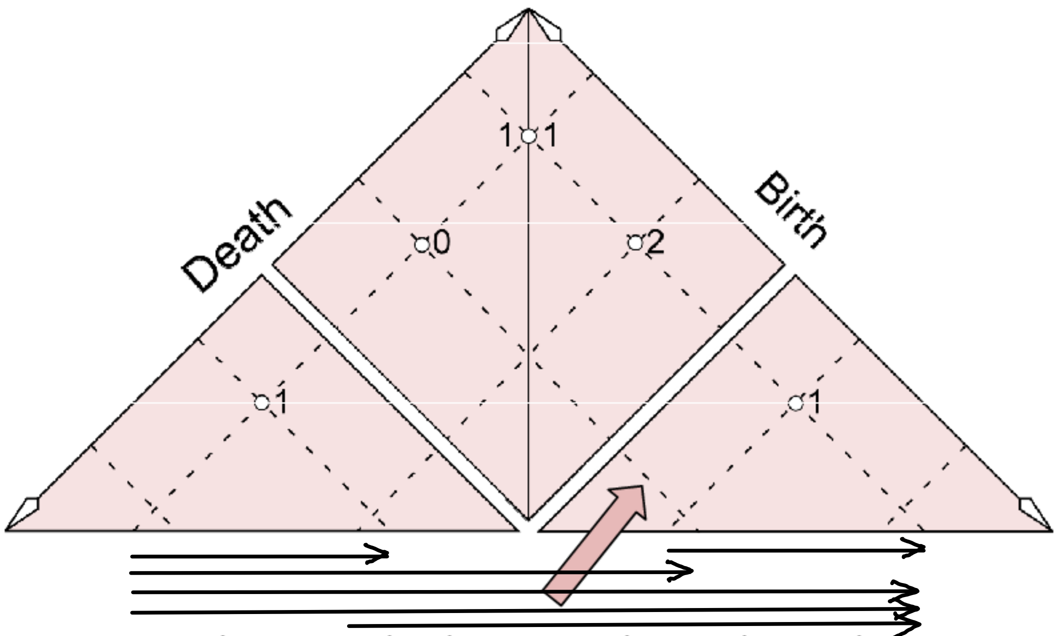


When we represent the image using cubical complex, we fail to demonstrate the symmetry in the diagram dictated by ~~Alexander~~ duality.

Lefschetz



When we add open faces of the blue cubes to the second covering at f=1, we create a cycle. It is destroyed when all cubes are in the second covering, at f=0.



Lower right triangle depicts relative homology classes: they are created and destroyed on the way up

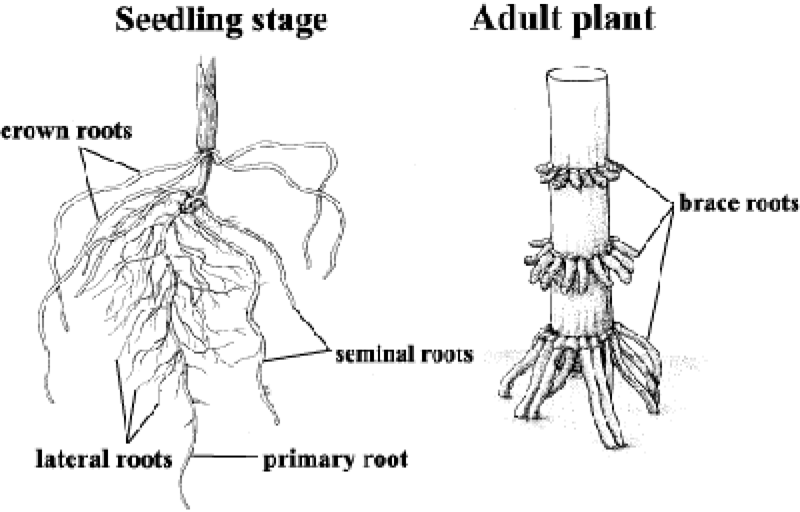
Part I The Difficulty

Part II The Motivation

Part III The Solution

I.1

Major root types of maize



drawings by: Miwa Kojima, Schnable lab, ISU

II.2

Understanding Genetic Basis of Root Traits

Genetic sequence of agricultural plants: rice and maize

Root traits

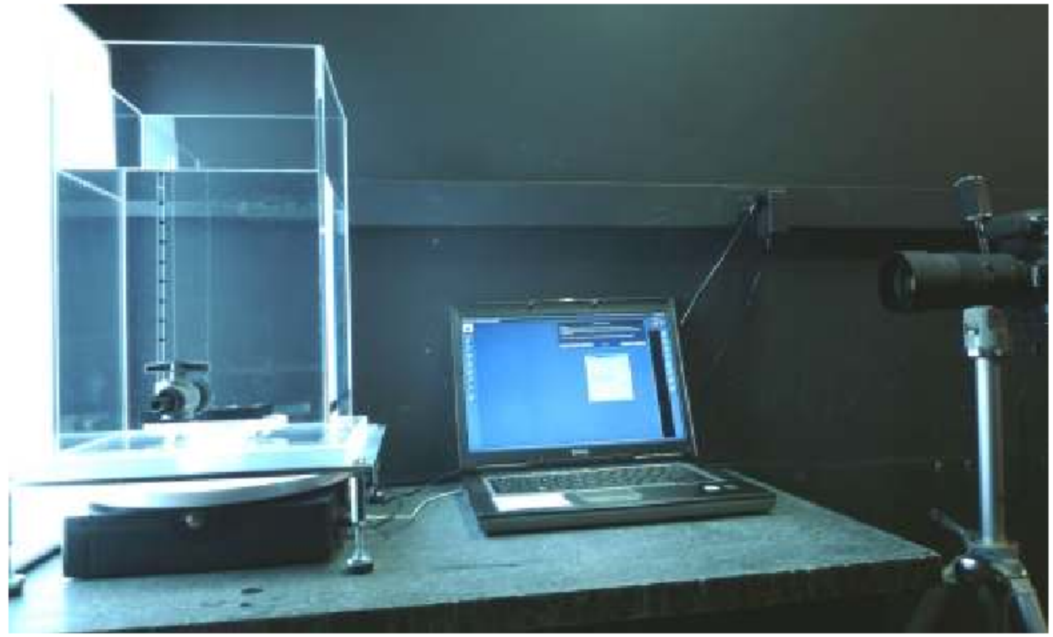
```
                                  *          20          *
gi|1992974 : -------ATACTTACCTGGACGGGGTCAACGGGC
```

II.3 Root System Description

- Global Traits: depth, volume, surface area, length
- Branching statistics: branching frequency, root types and hierarchy, length and width per root branch
- Dynamics traits: elongation, branching rates
- Local traits: related to the response of a plant to localized nutrients.

II.4 # Root Imaging Setup

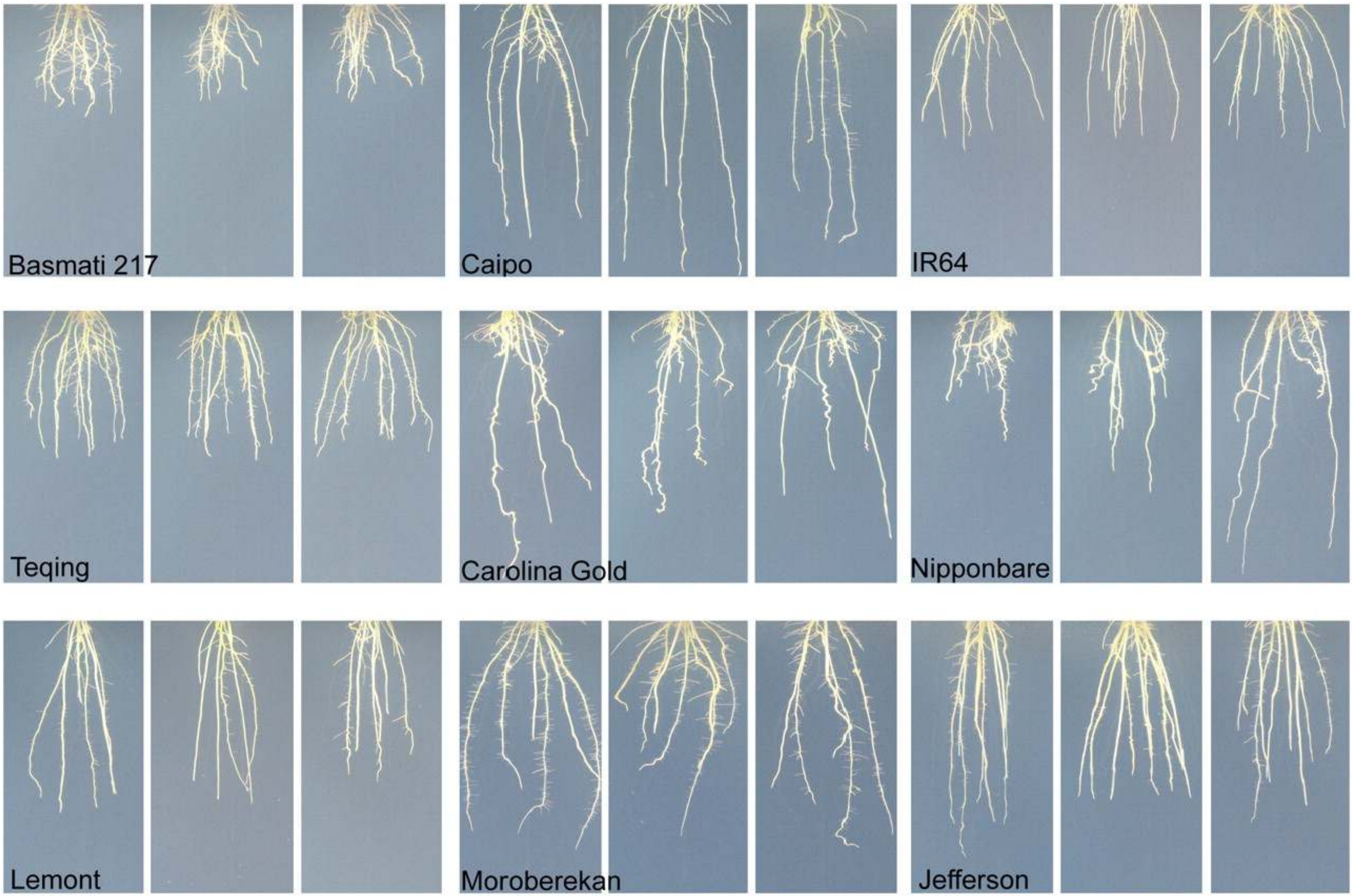
Several rice and maize genotypes grow in gel for 2-3 weeks



Automated imaging setup

II.5

Root Architecture Variation



II.6

3D Root Reconstruction

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3D Root Reconstruction

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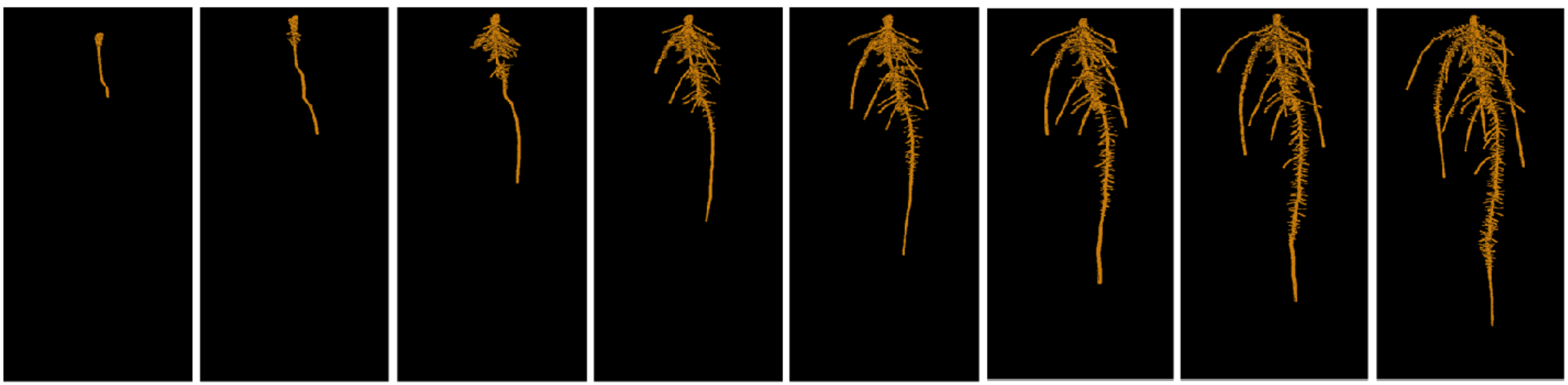
3D Root Reconstruction

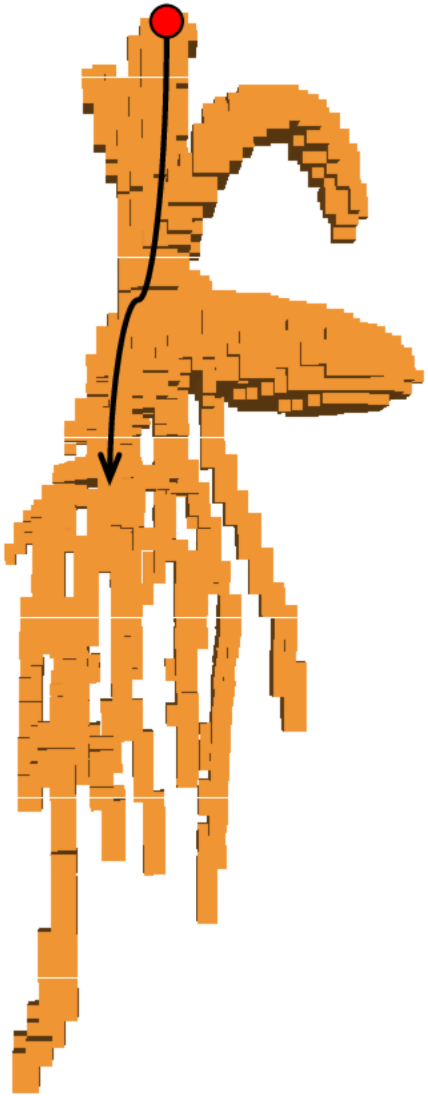
II.7 Analysis of Time-Series 3D Data

Sequences of 3D root reconstructions in time



day 1 day 2 day 3 day 4 day 5 day 6 day 7 day 8

II.8 Analysis of Shape of Root Systems



Geodesic function from the top of the root.

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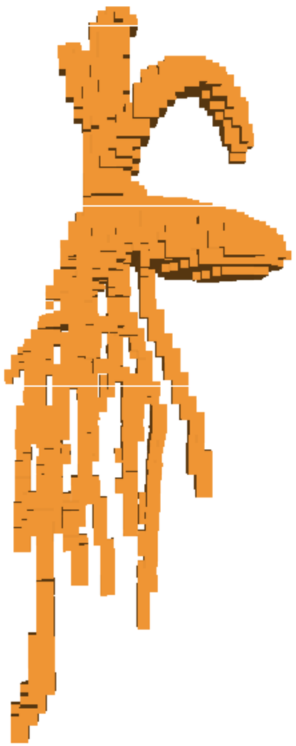
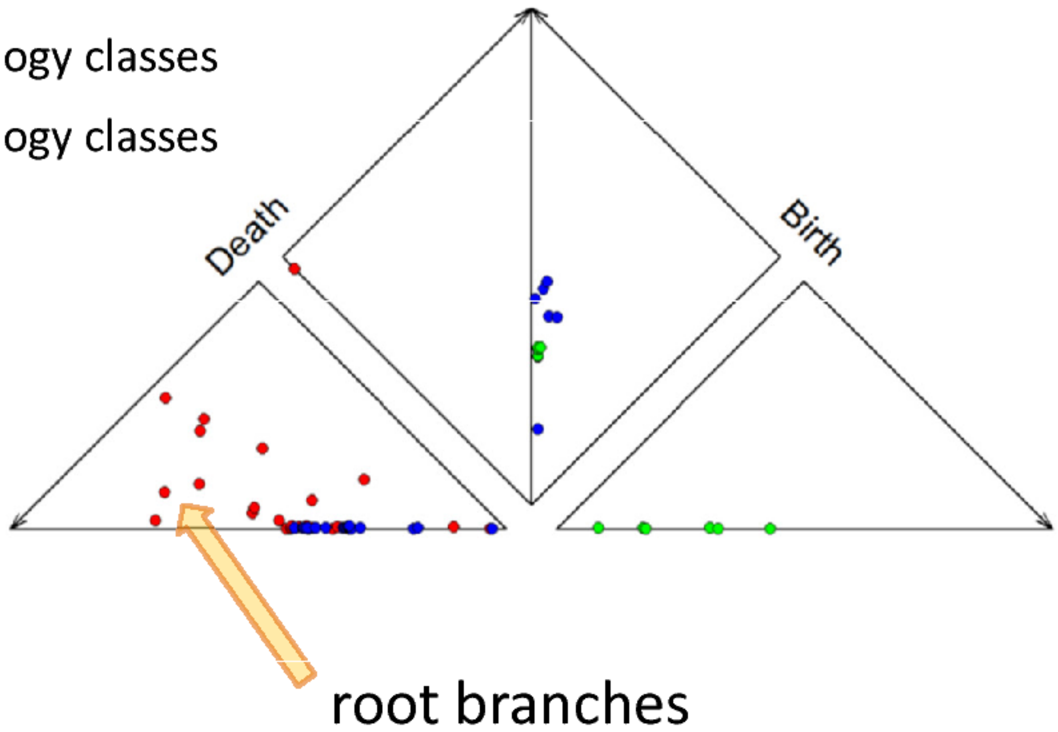


- 0d homology classes
- 1d homology classes
- 2d homology classes



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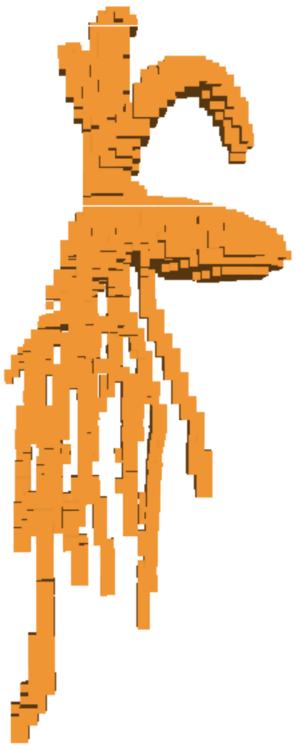

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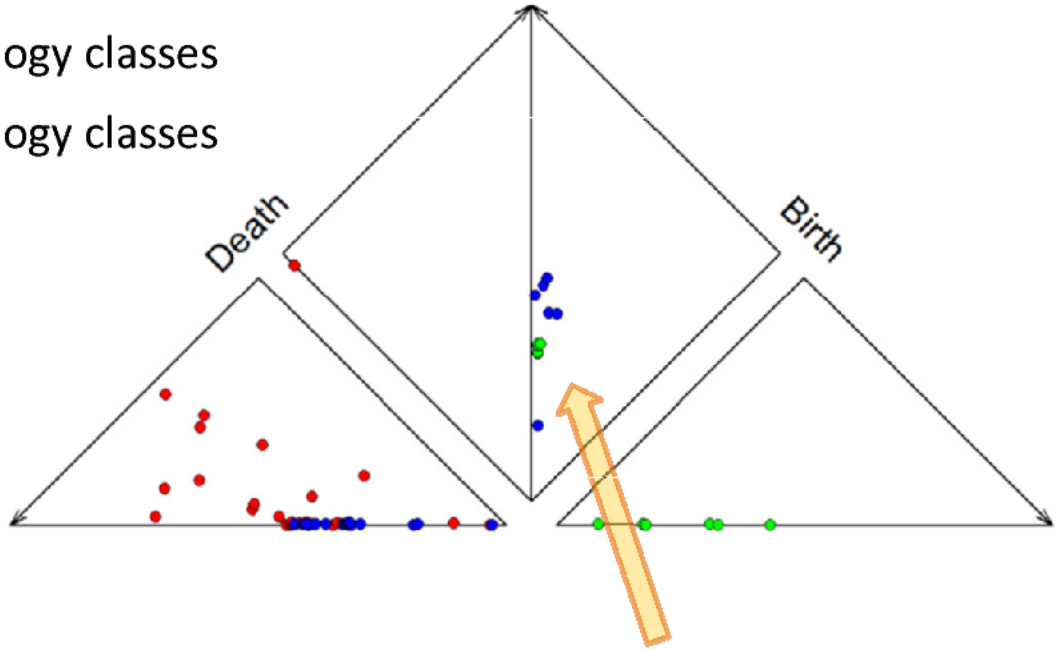
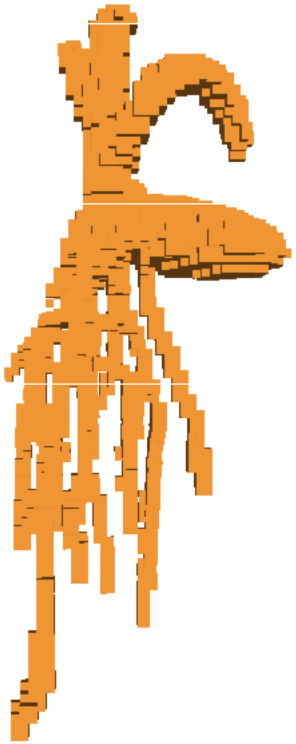


loops and voids due to artifacts of the shape acquisition process

Ⅱ.8

Analysis of Shape of Root Systems



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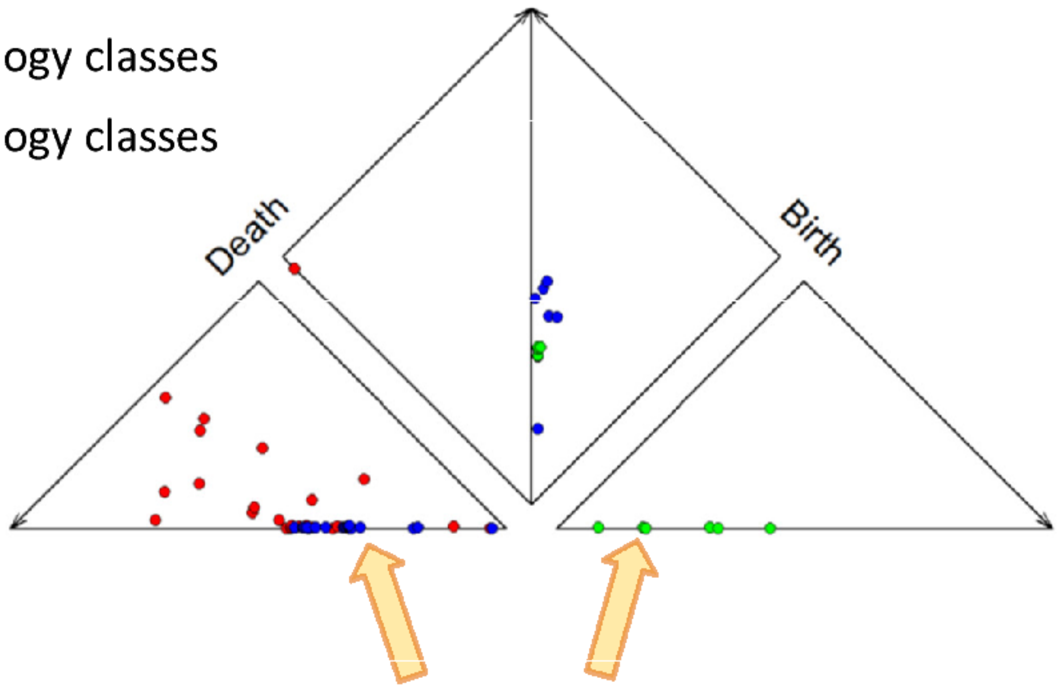

ceiling and floor depressions

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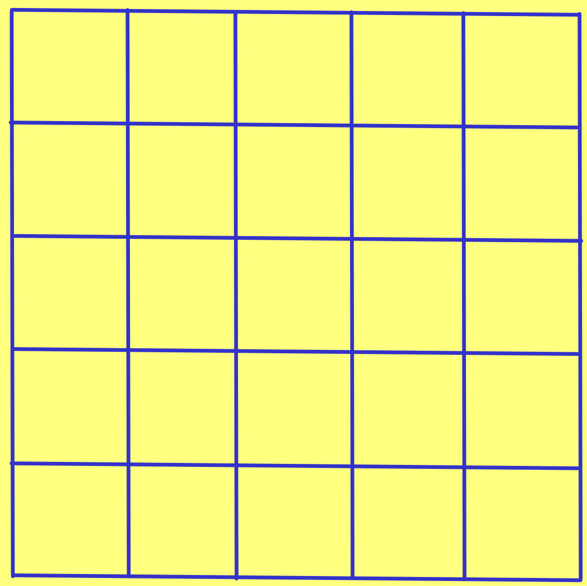
III.1 Digital Images



III.1 Digital Images

5	6	8	10	11
4	24	25	19	9
2	23	18	20	7
3	17	22	21	12
1	16	15	14	13

$$f : \mathcal{U}_2 \to \mathbb{R}$$

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5	6	8	10	11
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$$f : \mathcal{U}_2 \to \mathbb{R}$$

$$\varphi_r(U) = \begin{cases} 1 & \text{if } f(U) \geq r \\ 0 & \text{if } f(U) < r \end{cases}$$

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foreground $\varphi_r^{-1}(1)$

background $\varphi_r^{-1}(0)$

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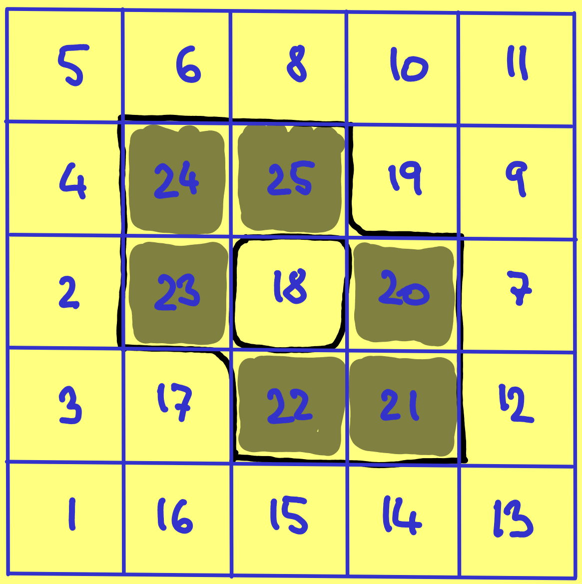


$r = 19.5$

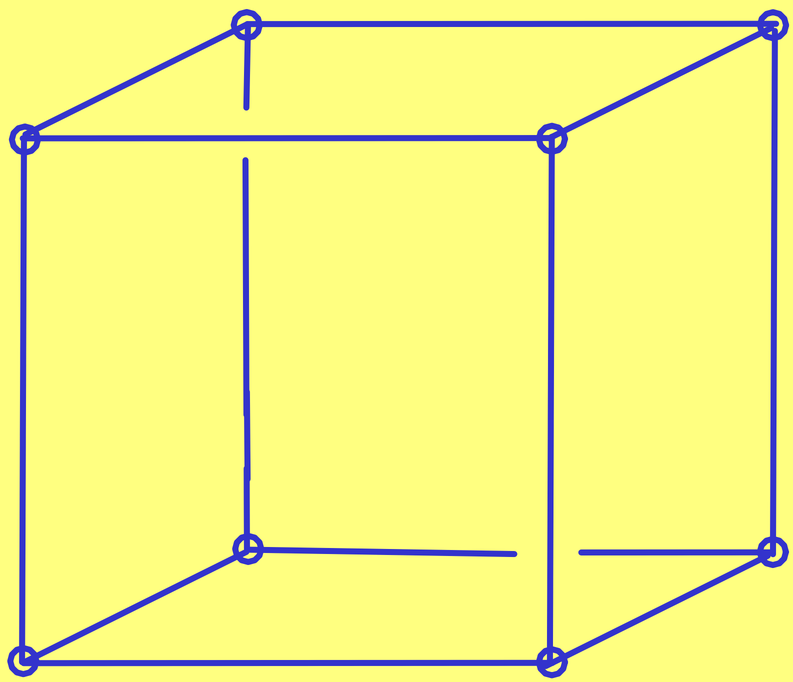
$f : \mathcal{U}_2 \to \mathbb{R}$

$$\varphi_r(U) = \begin{cases} 1 & \text{if } f(U) \geq r \\ 0 & \text{if } f(U) < r \end{cases}$$

foreground $\varphi_r^{-1}(1)$

background $\varphi_r^{-1}(0)$

III.2 Dual Complex



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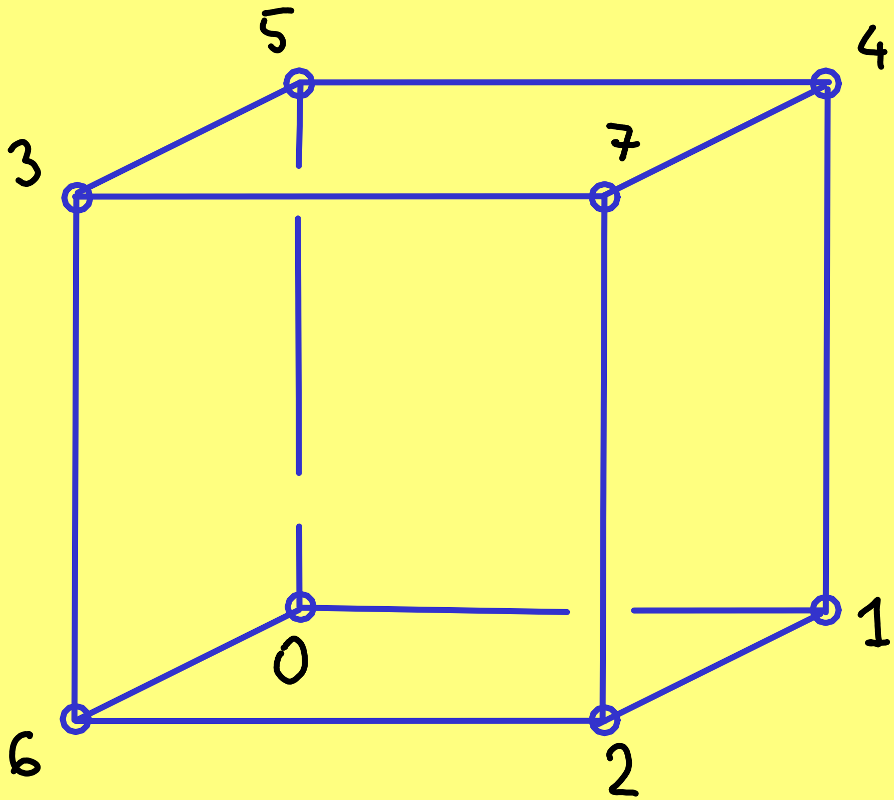

$\mathcal{D} \subseteq 2^{\mathcal{U}_n}$ obtained by inductive starring

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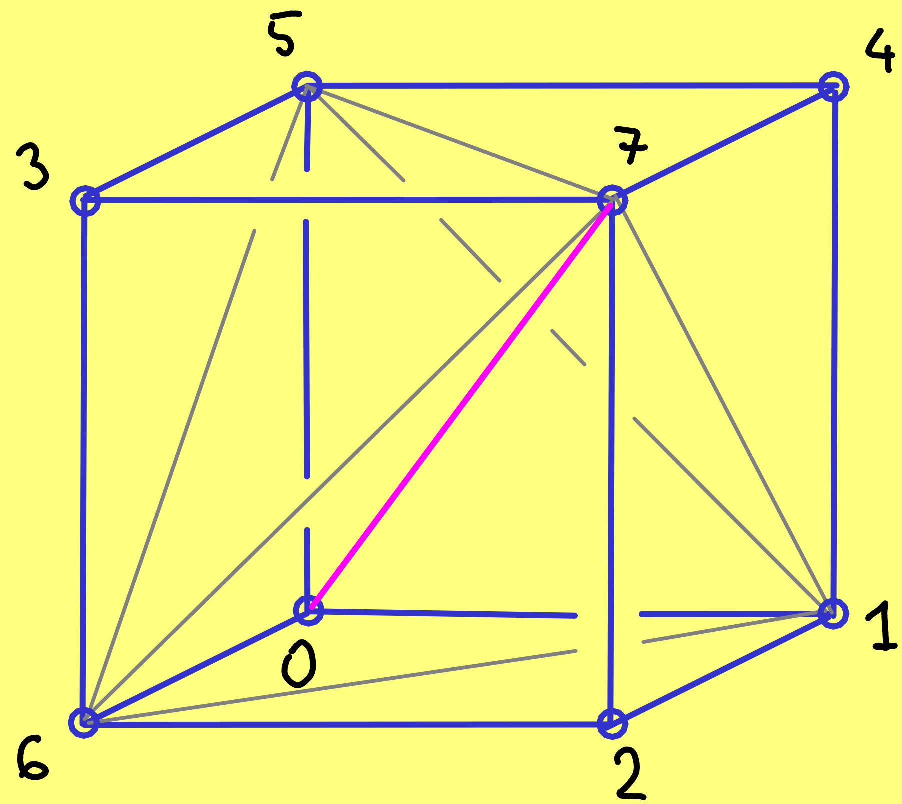

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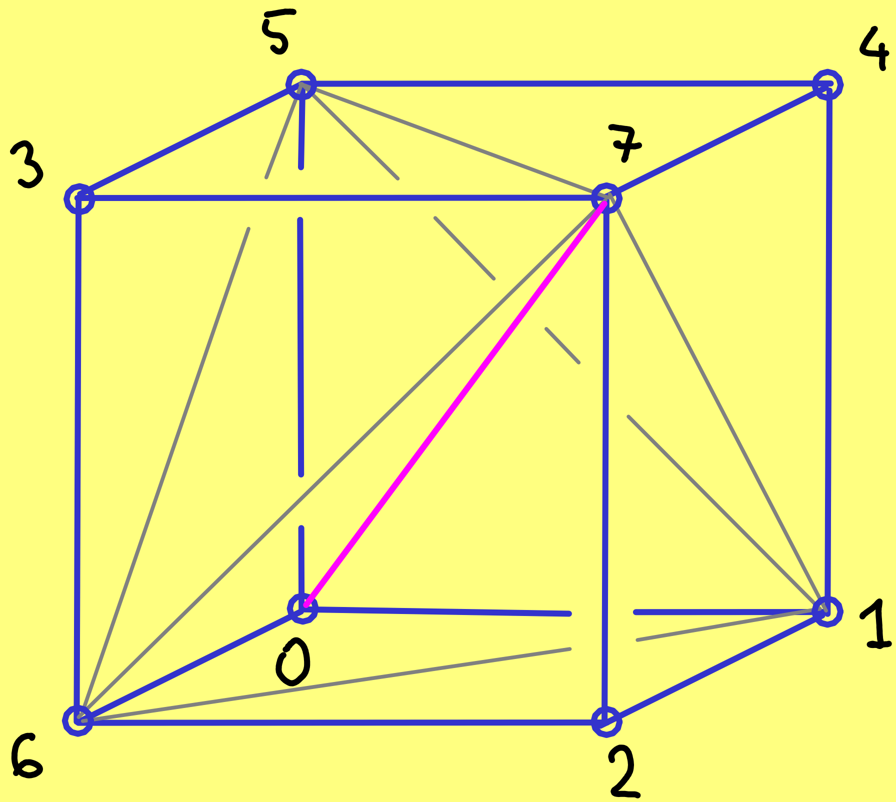

$D \subseteq 2^{U_n}$ obtained by inductive starring

$\mathbb{D} = |D|$

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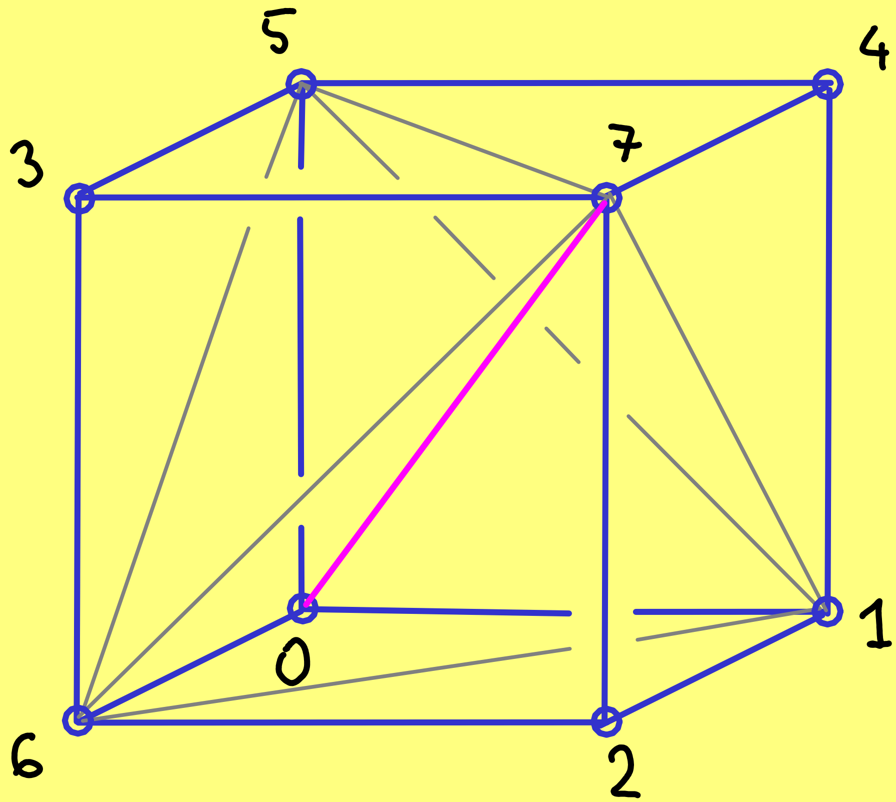

$\mathcal{D} \subseteq 2^{\mathcal{U}_n}$ obtained by inductive starring

$\mathbb{D} = |\mathcal{D}|$

$\delta : \mathbb{D} \to \mathbb{R}$

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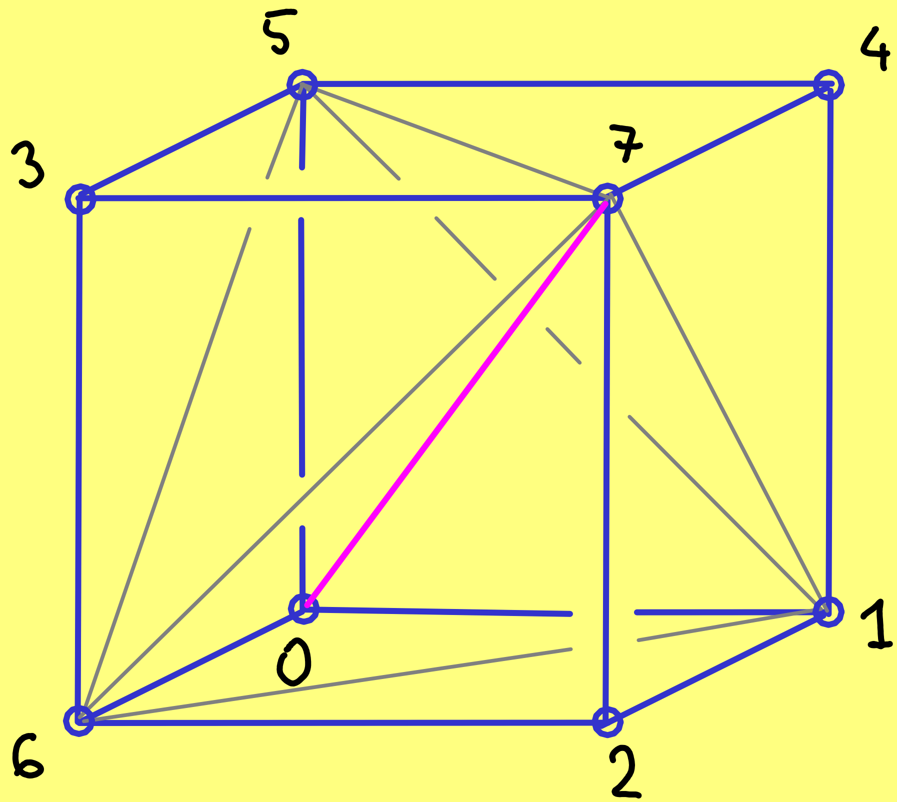

$\mathbb{D} \subseteq 2^{U_n}$ obtained by inductive starring

$D = |\mathbb{D}|$

$\delta : \mathbb{D} \to \mathbb{R}$

THM. $\mathrm{Dgm}(\delta)$ is consistent w. close fore- open backgrounds.

III.3 COMPUTATION

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D^i is full subcomplex of i first vertices

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$\mathcal{D}^i$ ($\mathcal{D}_i$) is full subcomplex of i first (last) vertices

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$$\ldots \to H(D^i) \to \ldots \to H(D, D_i) \to \ldots$$

give same diagram.

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$\mathcal{D}^i$ $(\mathcal{D}_i)$ is full subcomplex of i first (last) vertices

$$\begin{array}{ccccccccc}
\cdots & \to & H(\mathbb{D}^i) & \to & \cdots & \to & H(\mathbb{D}, \mathbb{D}_i) & \to & \cdots \\
 & & \downarrow & & & & \downarrow & & \\
\cdots & \to & H(\mathcal{D}^i) & \to & \cdots & \to & H(\mathcal{D}, \mathcal{D}_i) & \to & \cdots
\end{array}$$

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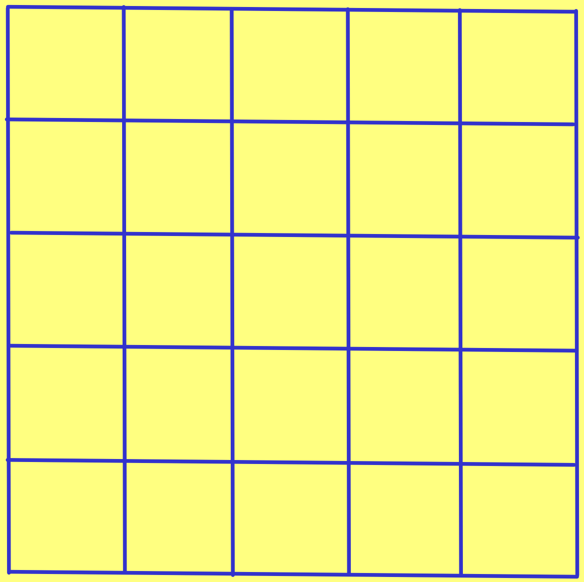
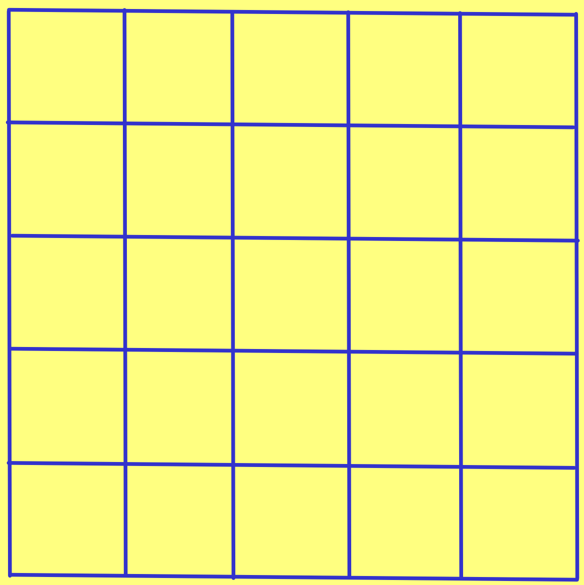
 & & \big\downarrow \text{iso} & & & & \big\downarrow \text{iso} & & \\
\cdots & \to & H(\mathcal{D}^i) & \to & \cdots & \to & H(\mathcal{D}, \mathcal{D}_i) & \to & \cdots
\end{array}$$

give same diagram.

III.4 Weighted Voronoi Tessellation



III.4 Weighted Voronoi Tessellation



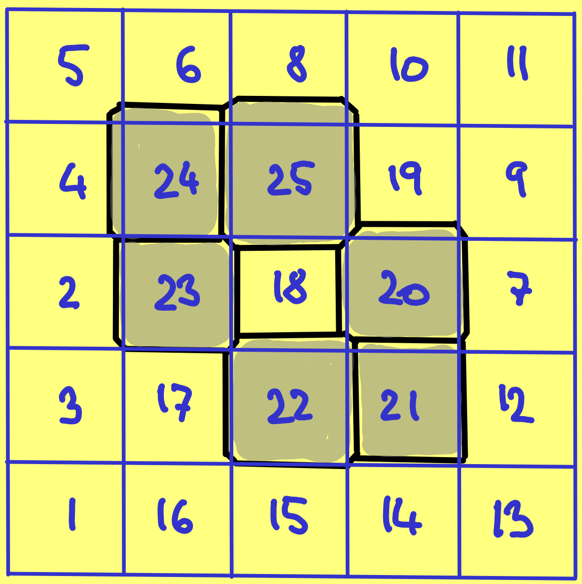
$$\|x - v_i\|^2 - \omega_i \leq \|x - v_j\|^2 - \omega_j \quad \forall j$$

III.4 Weighted Voronoi Tessellation

5	6	8	10	11
4	24	25	19	9
2	23	18	20	7
3	17	22	21	12
1	16	15	14	13

$$\|x - v_i\|^2 - \omega_i \leq \|x - v_j\|^2 - \omega_j \quad \forall j$$

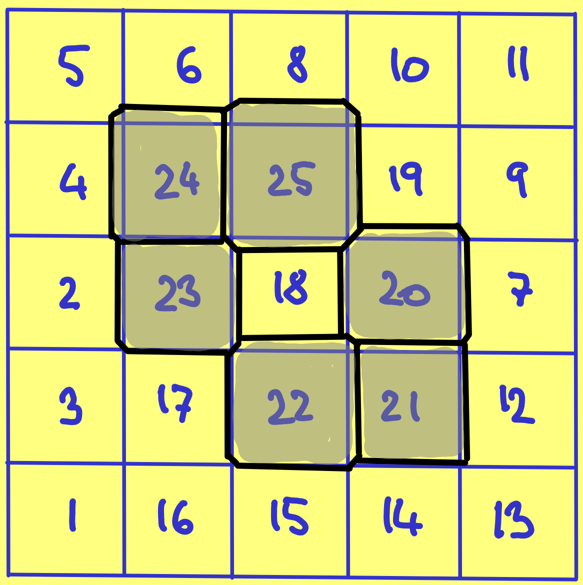
III.4 Weighted Voronoi Tessellation



$$\|x - v_i\|^2 - \omega_i \leq \|x - v_j\|^2 - \omega_j \quad \forall j$$

III.4 WEIGHTED VORONOI TESSELLATION

(generically simple)



$$\|x - v_i\|^2 - \omega_i \leq \|x - v_j\|^2 - \omega_j \qquad \forall j$$

III.5 PRIMITIVES

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$$\det \Delta = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} \\ 1 & x_{2,1} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1,1} & \cdots & x_{n+1,n} \end{vmatrix}$$

orientation test

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$$\det \Lambda = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} & \sum x_{1,j}^2 \\ 1 & x_{2,1} & \cdots & x_{2,n} & \sum x_{2,j}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{n+2,1} & \cdots & x_{n+2,n} & \sum x_{n+2,j}^2 \end{vmatrix}$$

in sphere test

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$$\det \Delta = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} \\ 1 & x_{2,1} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1,1} & \cdots & x_{n+1,n} \end{vmatrix}$$

orientation test

$$\det \Lambda = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} & \sum x_{1,j}^2 - \omega_1 \\ 1 & x_{2,1} & \cdots & x_{2,n} & \sum x_{2,j}^2 - \omega_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{n+2,1} & \cdots & x_{n+2,n} & \sum x_{n+2,j}^2 - \omega_{n+2} \end{vmatrix}$$

in sphere test

III.5 PRIMITIVES

$$\det \Delta = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} \\ 1 & x_{2,1} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1,1} & \cdots & x_{n+1,n} \end{vmatrix}$$

orientation test

$$\det \Lambda = \begin{vmatrix} 1 & x_{1,1} & \cdots & x_{1,n} & \sum x_{1,j}^2 - \omega_1 \\ 1 & x_{2,1} & \cdots & x_{2,n} & \sum x_{2,j}^2 - \omega_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{n+2,1} & \cdots & x_{n+2,n} & \sum x_{n+2,j}^2 - \omega_{n+2} \end{vmatrix}$$

in sphere test

$\omega_i = \varepsilon^i$ gives same triangulation for all $0 < \varepsilon < \frac{1}{(n+1)!+1}$

III.6 Proof of Correctness

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- $H(\mathbb{D}^a) \simeq H(\mathbb{U}^a)$ ← closed foreground

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- $H(\mathbb{D}^a) \simeq H(\mathbb{U}^a)$ ← closed foreground
- $H(\mathbb{D}, \mathbb{D}_a) \simeq H(\mathbb{U}, \mathbb{U}_a)$ ← open background
- $$\begin{array}{ccccccccc} \cdots & \to & H(\mathbb{D}^a) & \to & \cdots & \to & H(\mathbb{D}, \mathbb{D}_a) & \to & \cdots \\ & & \downarrow \text{iso} & & & & \downarrow \text{iso} & & \\ \cdots & \to & H(\mathbb{U}^a) & \to & \cdots & \to & H(\mathbb{U}, \mathbb{U}_a) & \to & \cdots \end{array}$$

THANKS TO

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lead by Philip Benfey @ Duke

THANK YOU