Computing all maps into a sphere

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Mathematics of collaboration:

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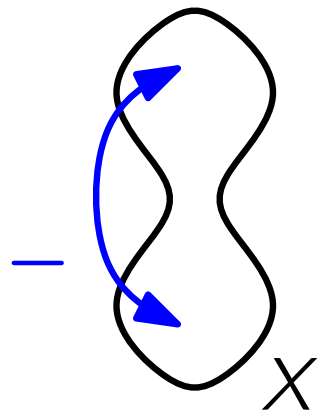

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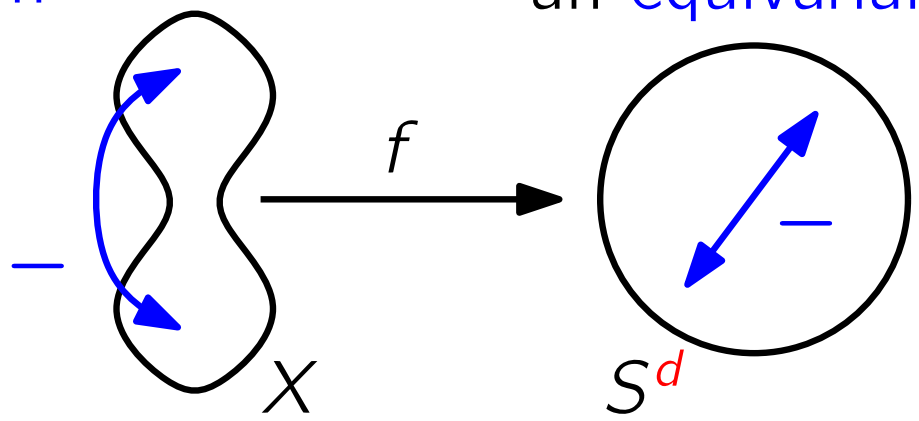

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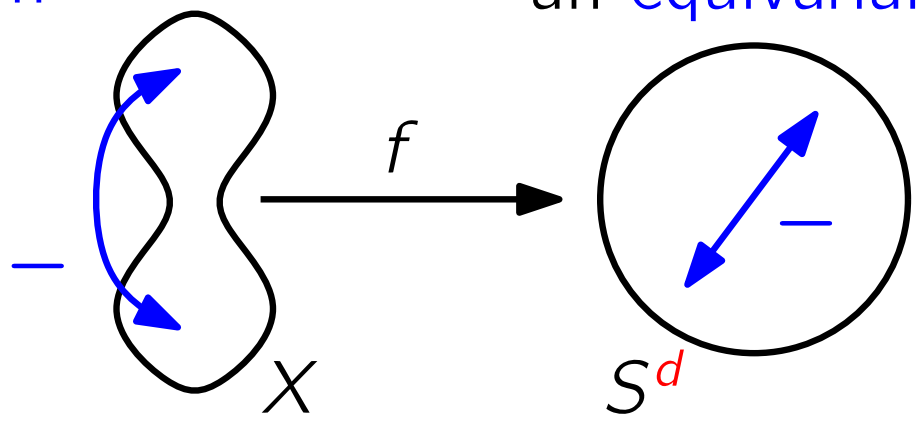

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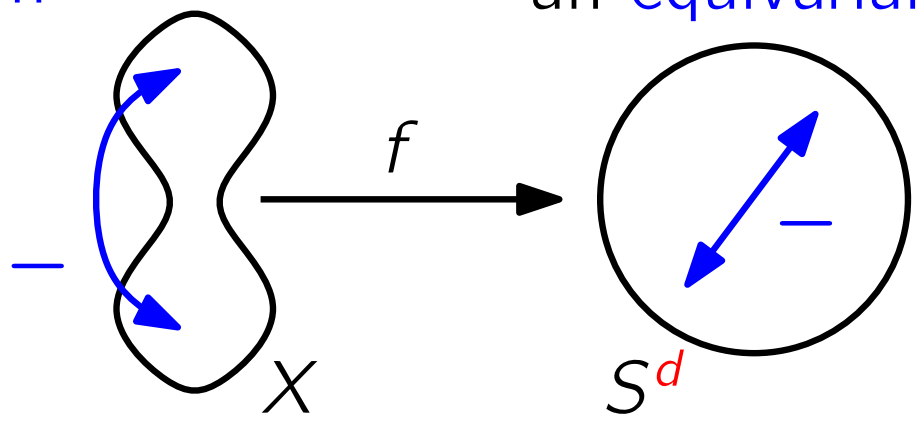

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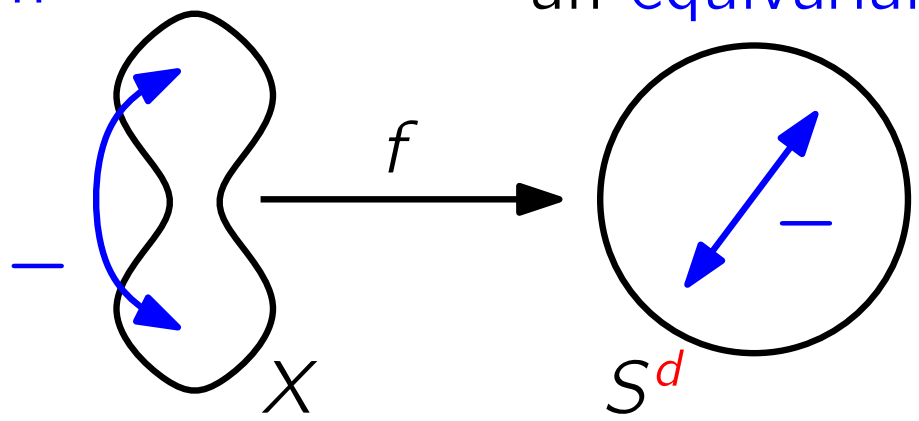

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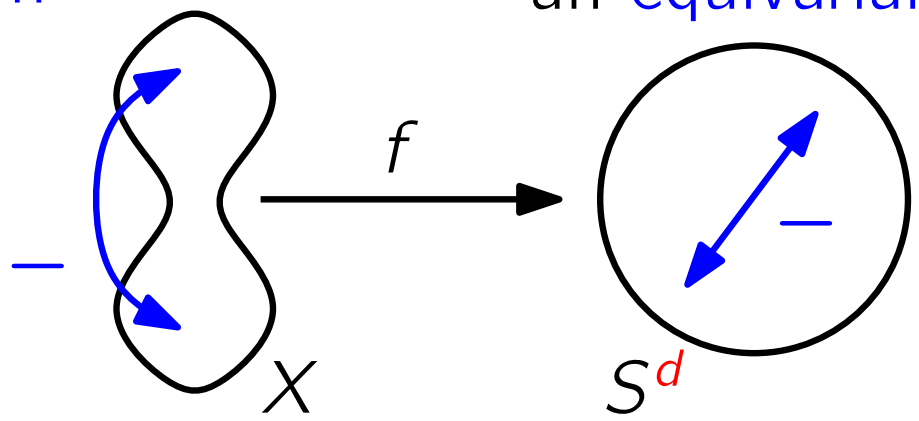

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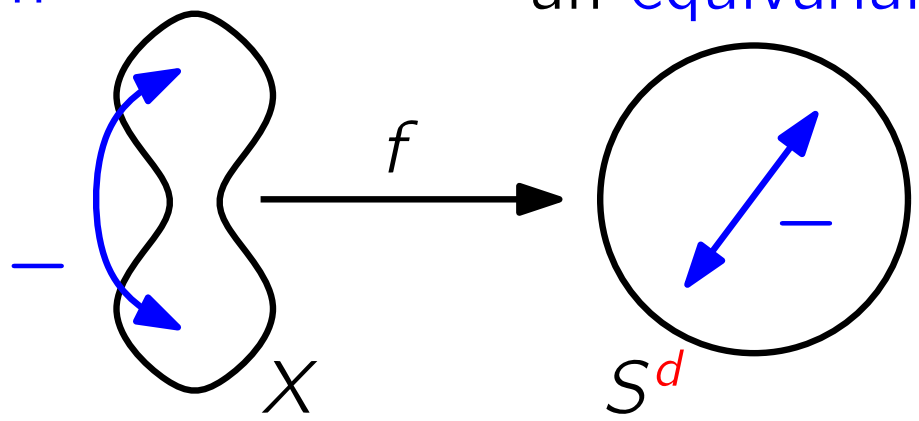

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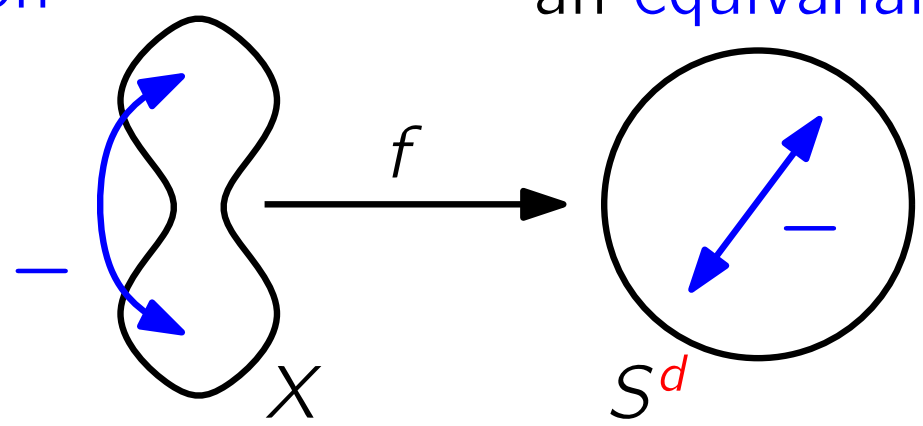

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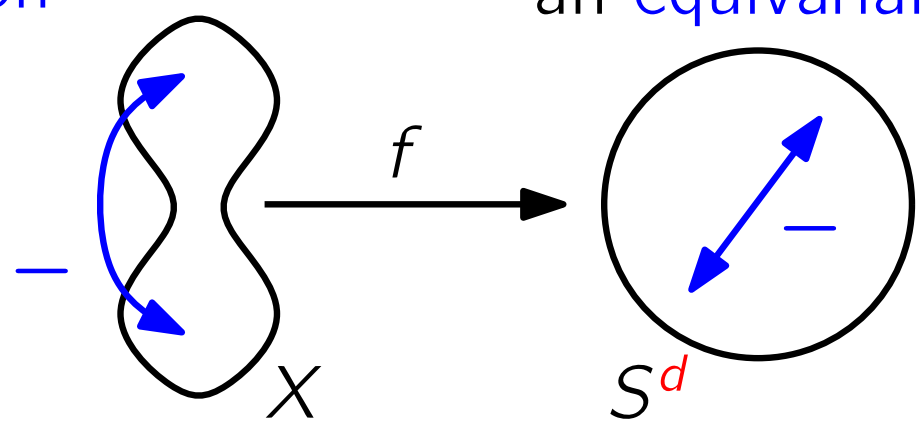

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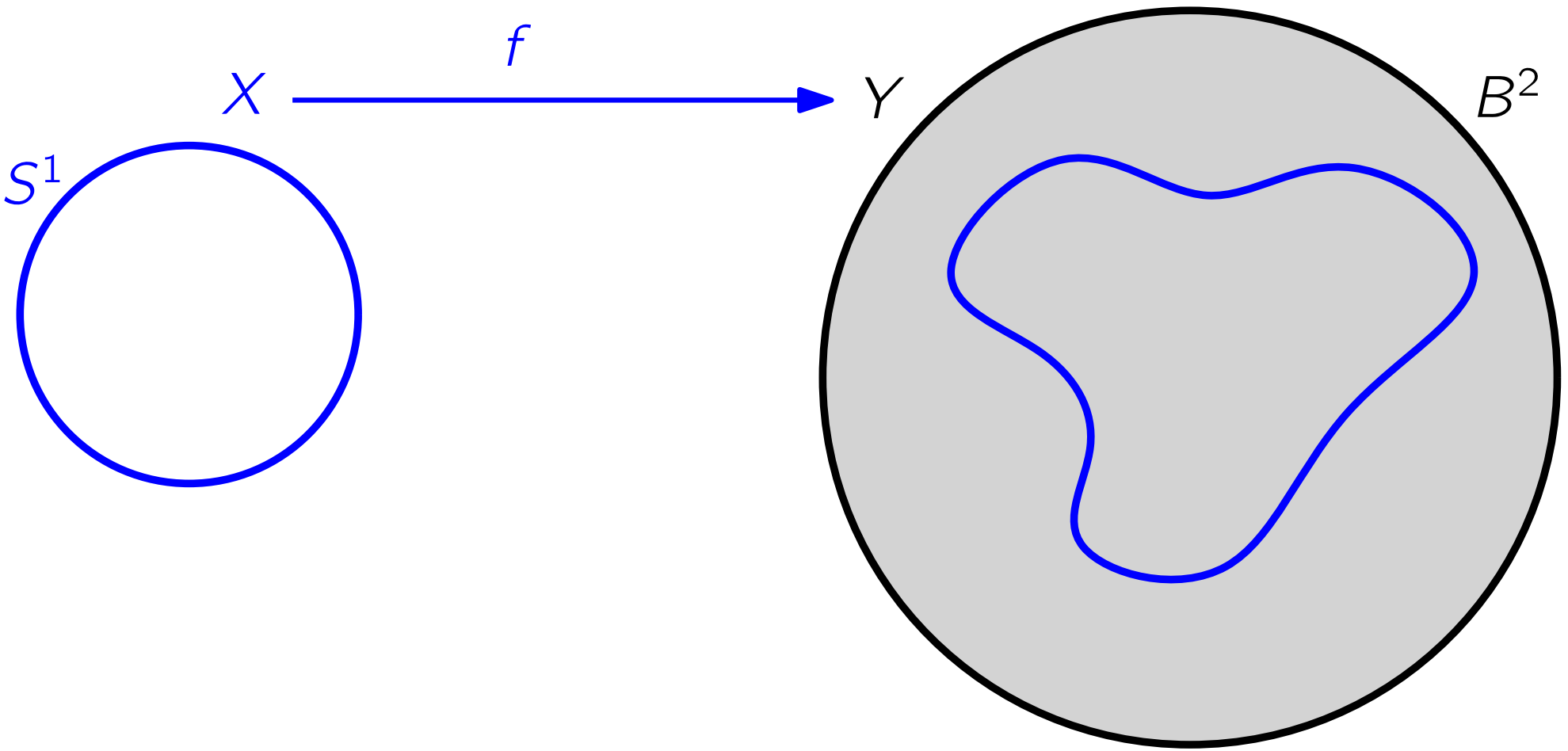

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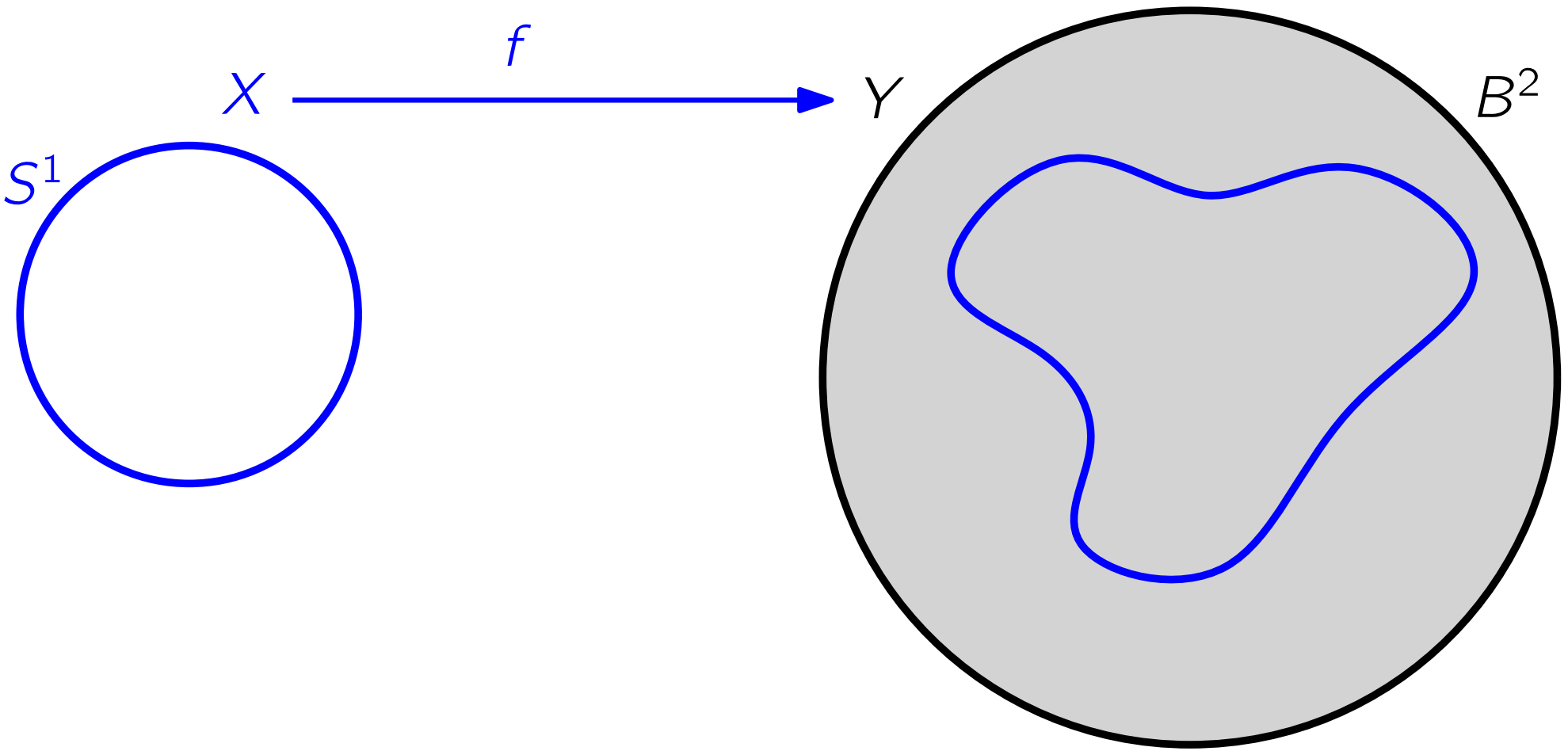

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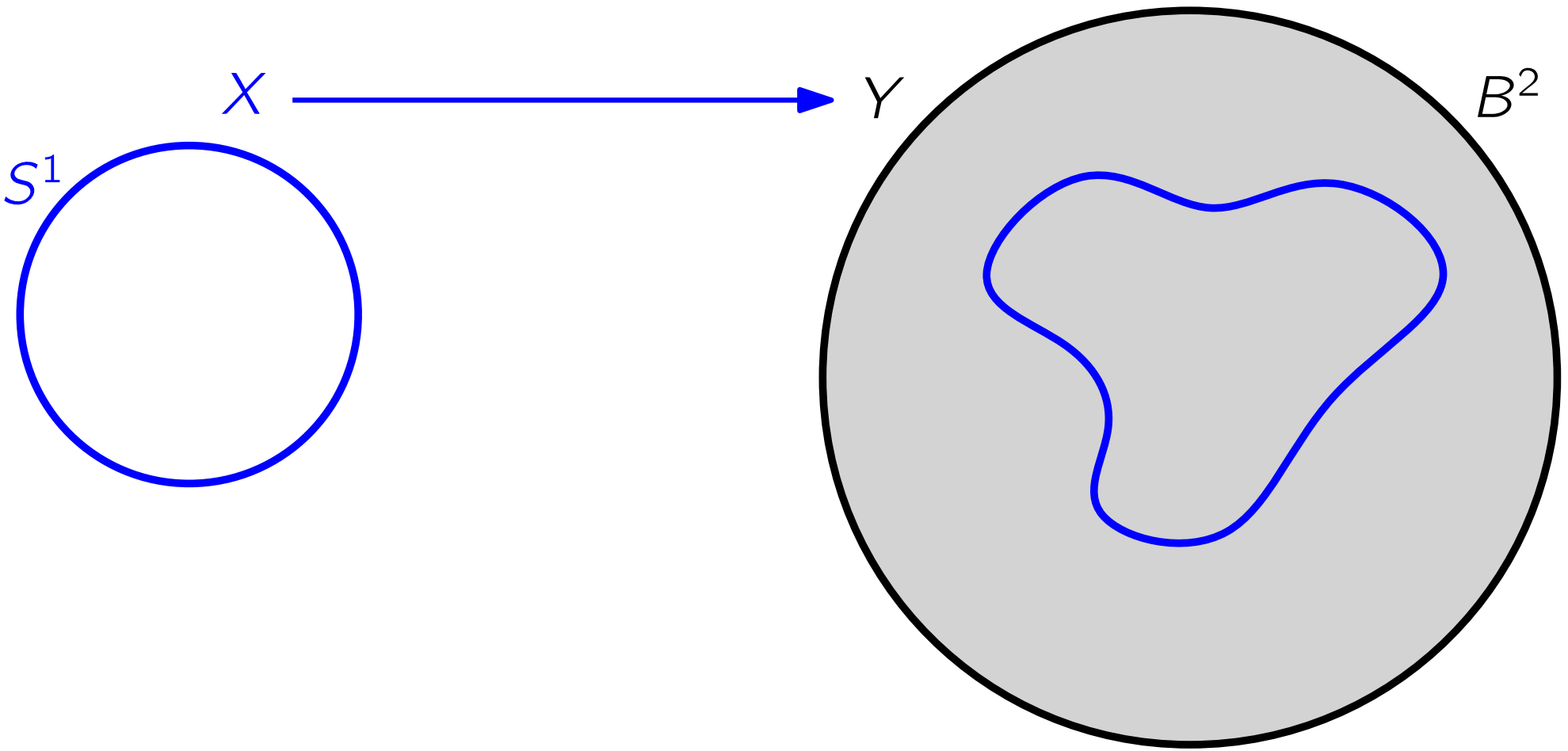

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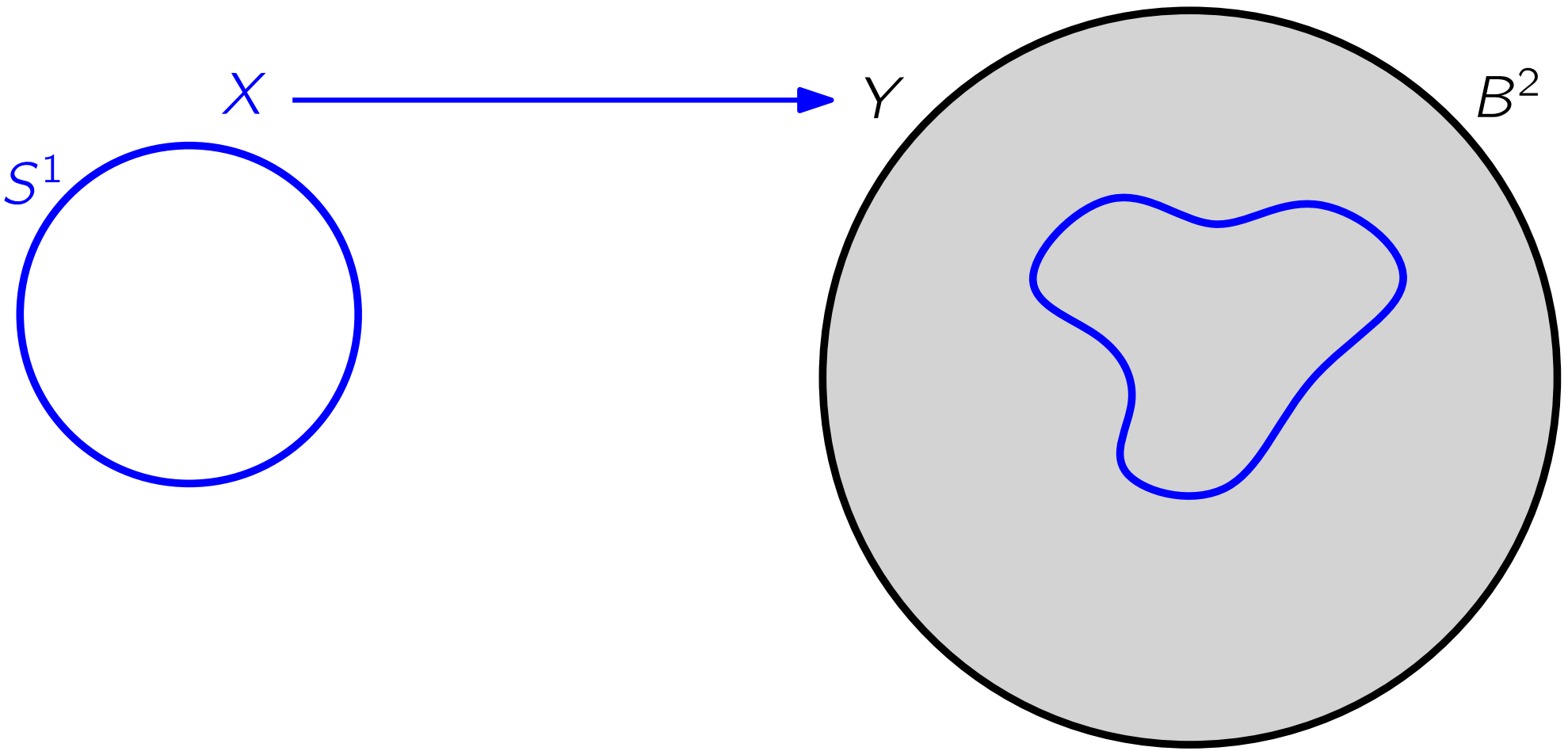

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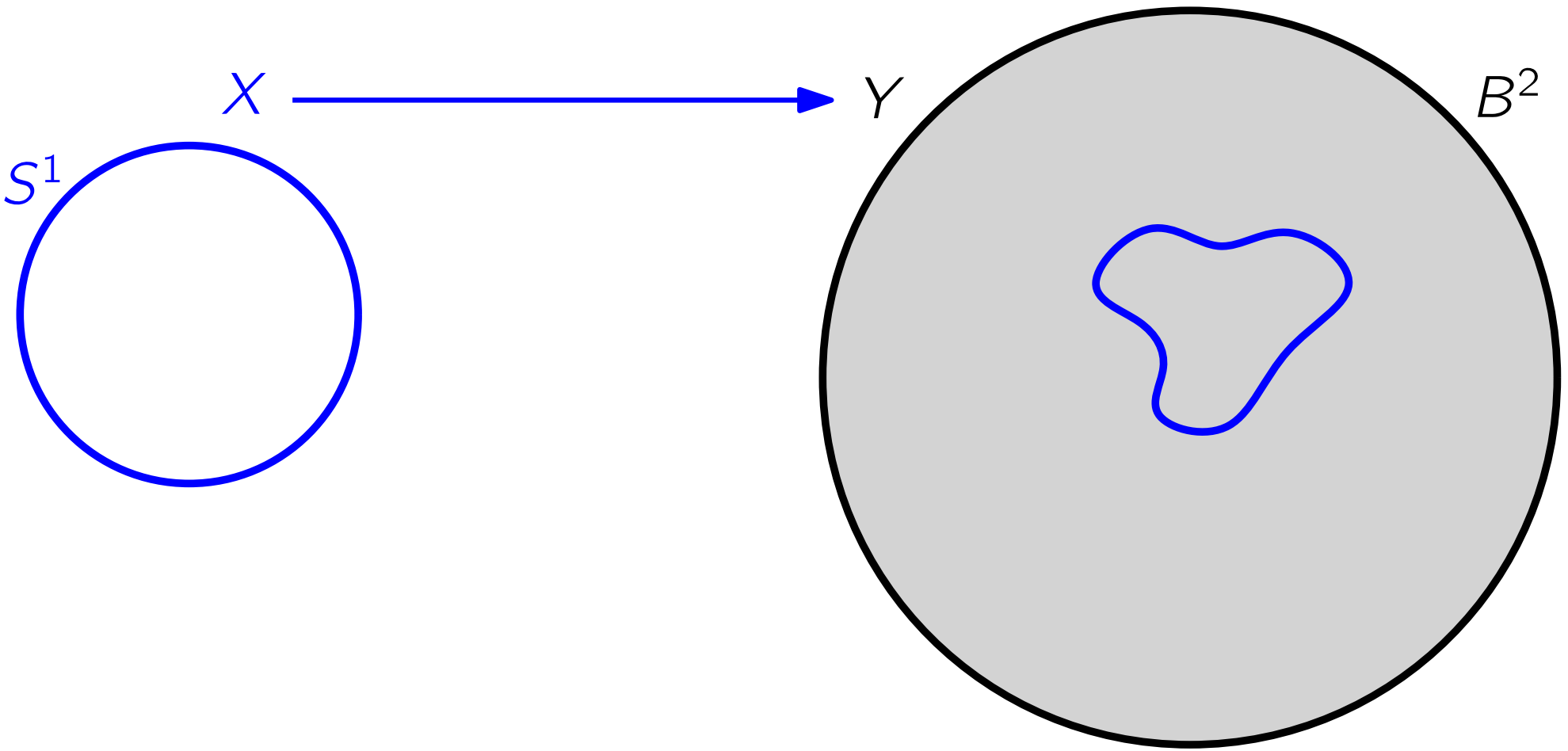

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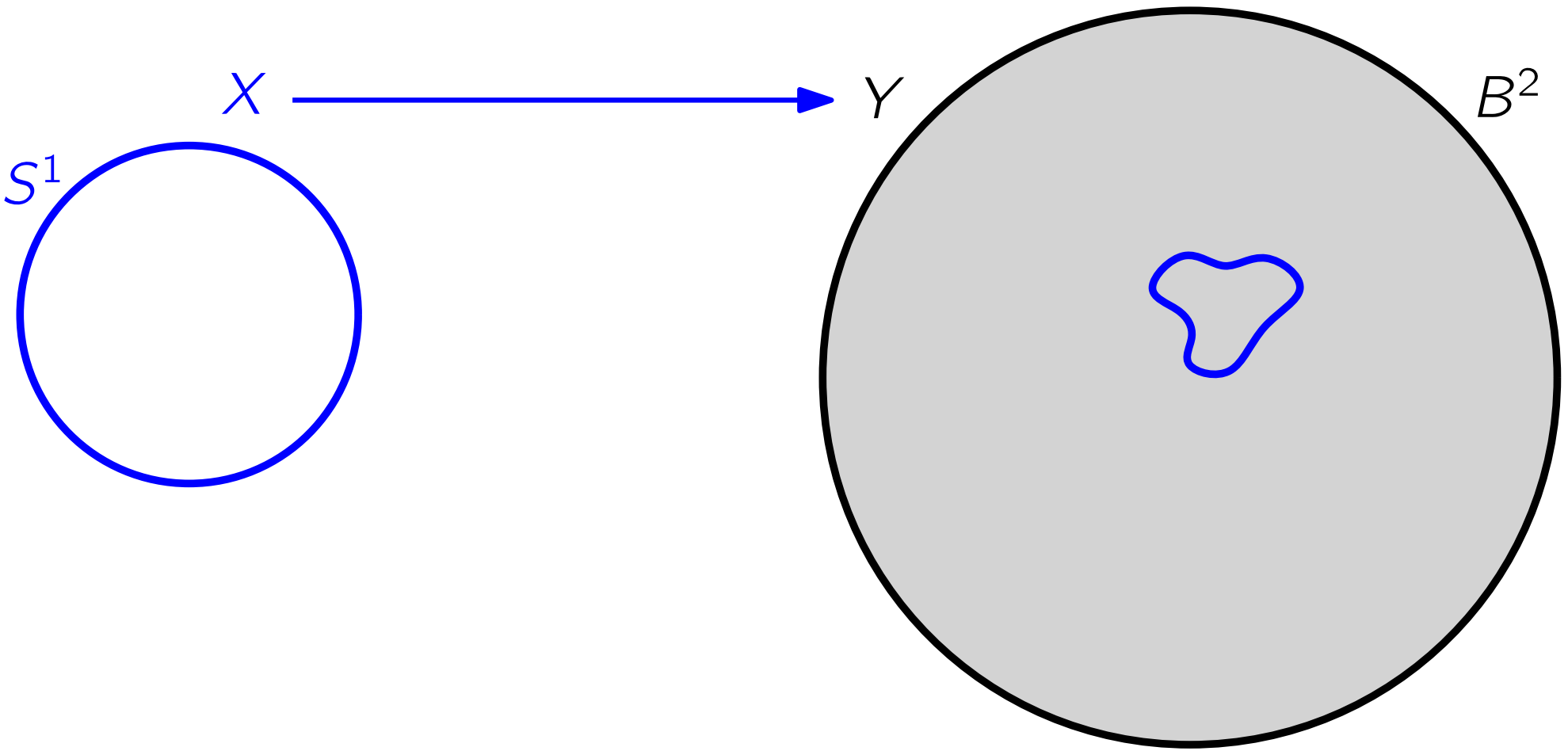

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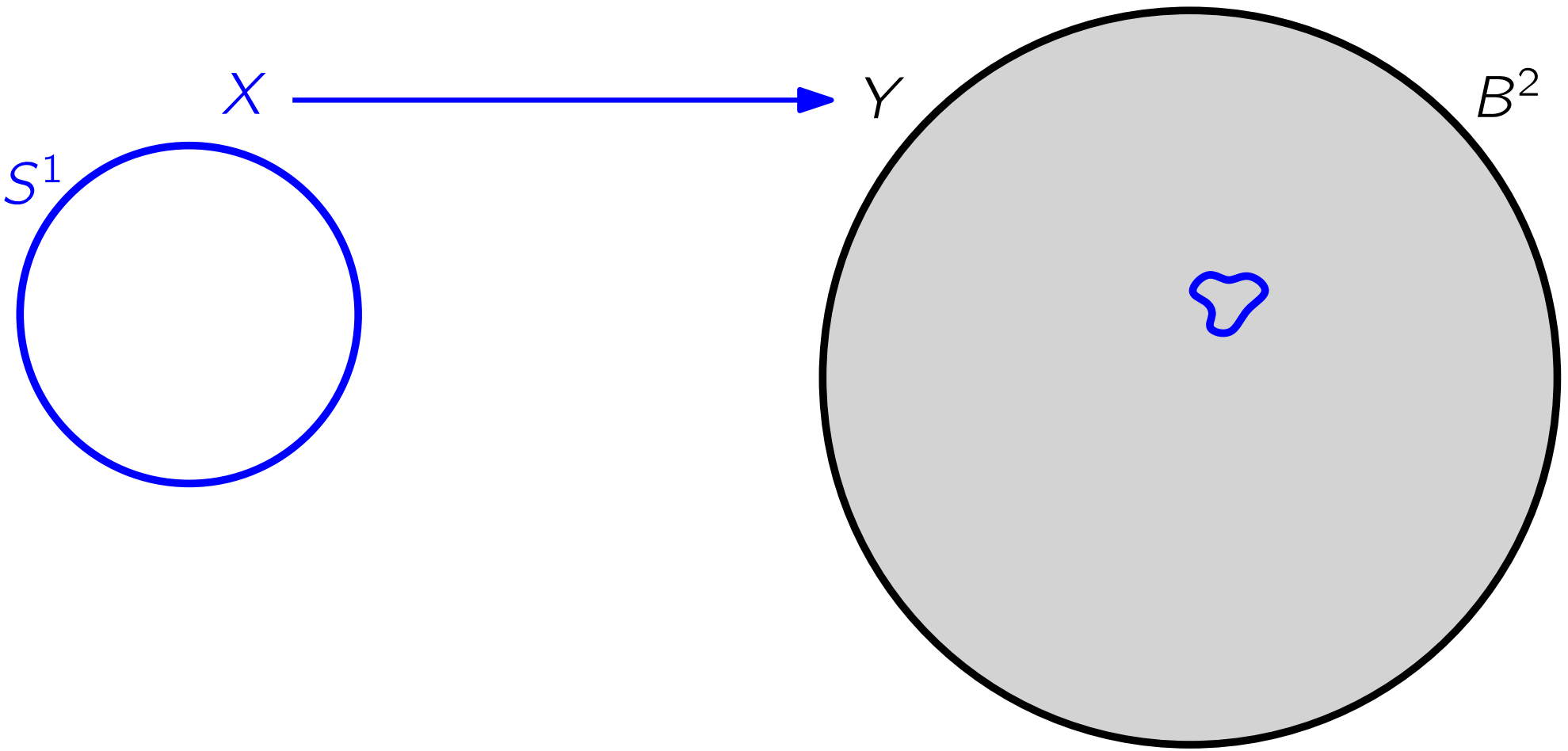

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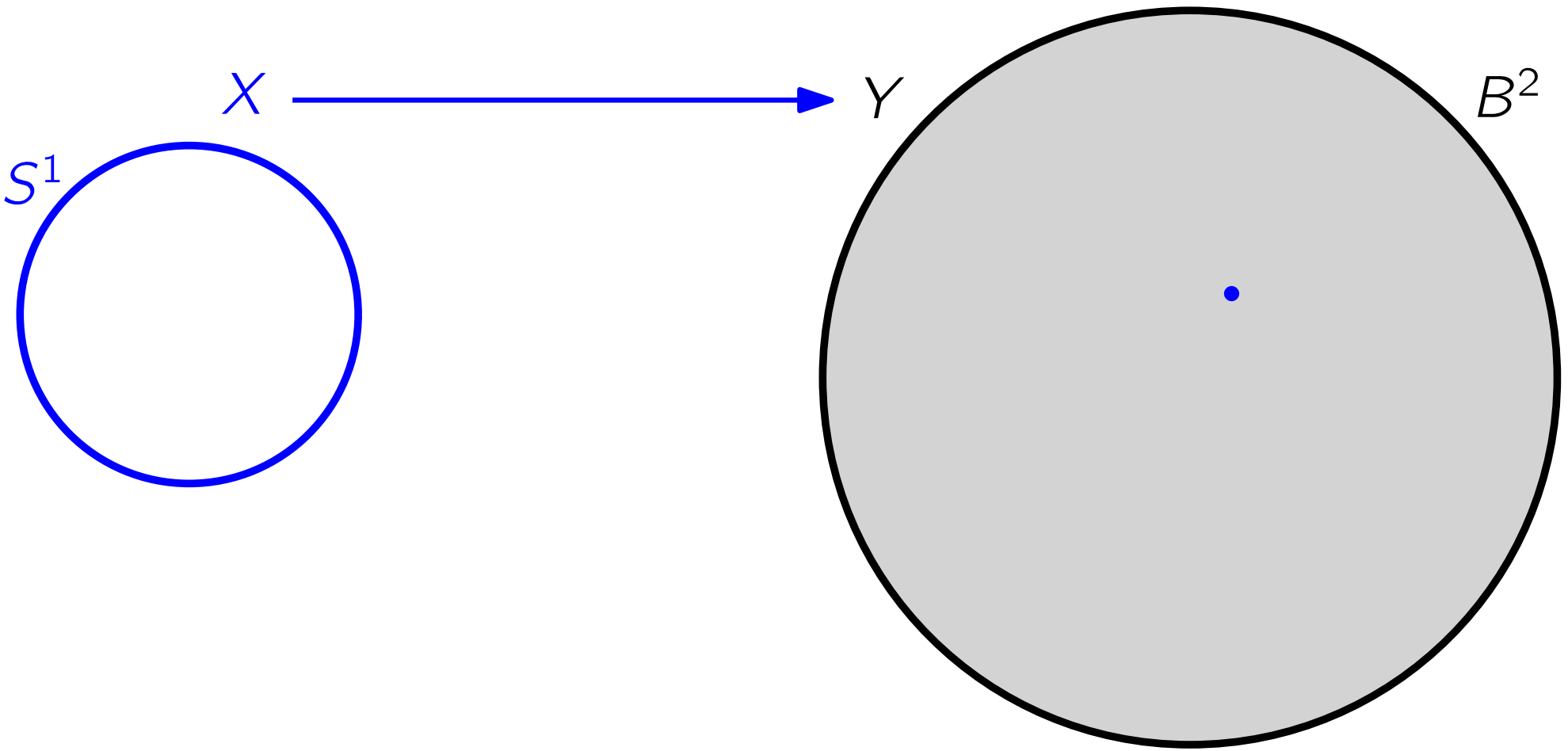

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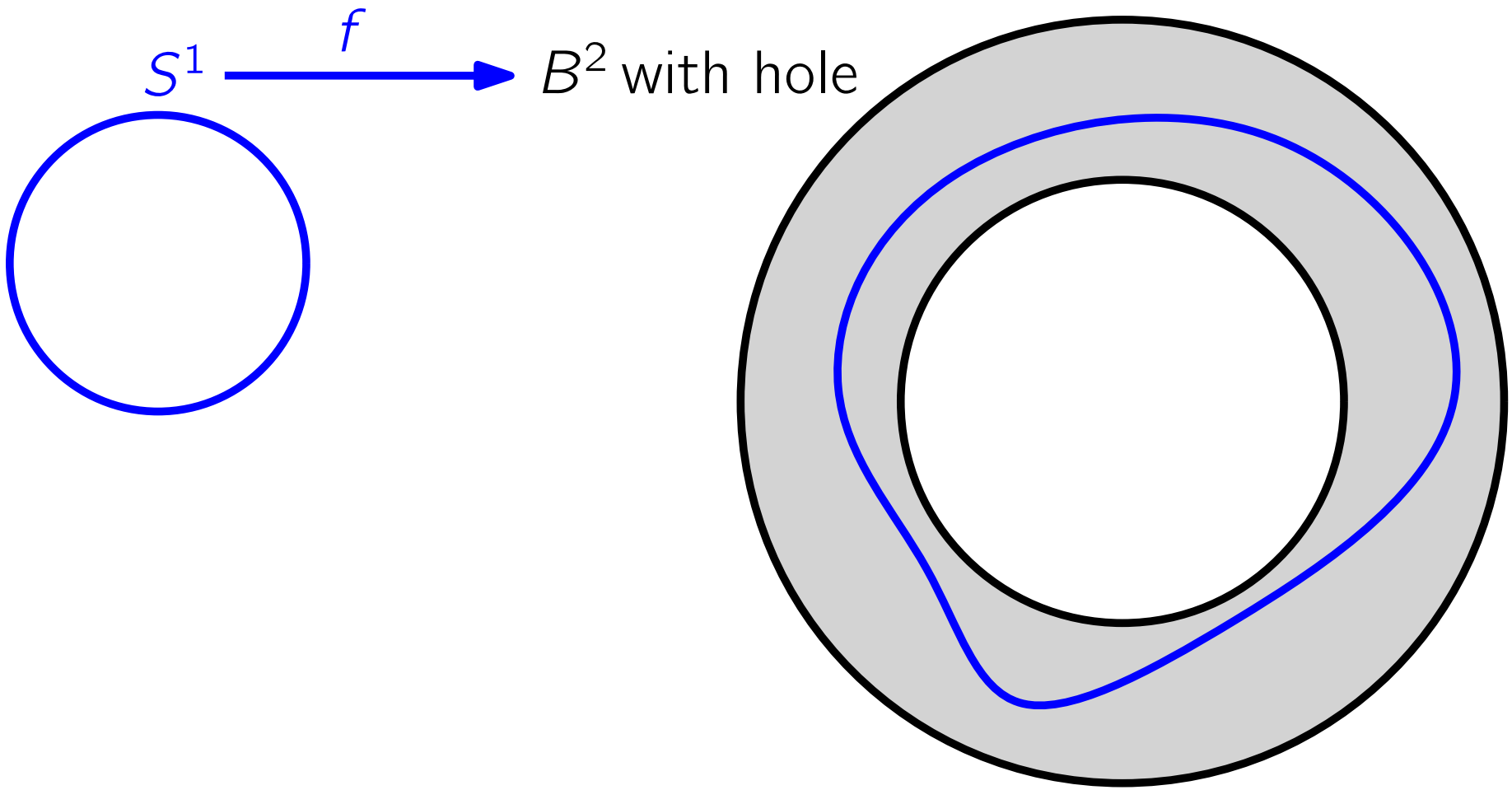

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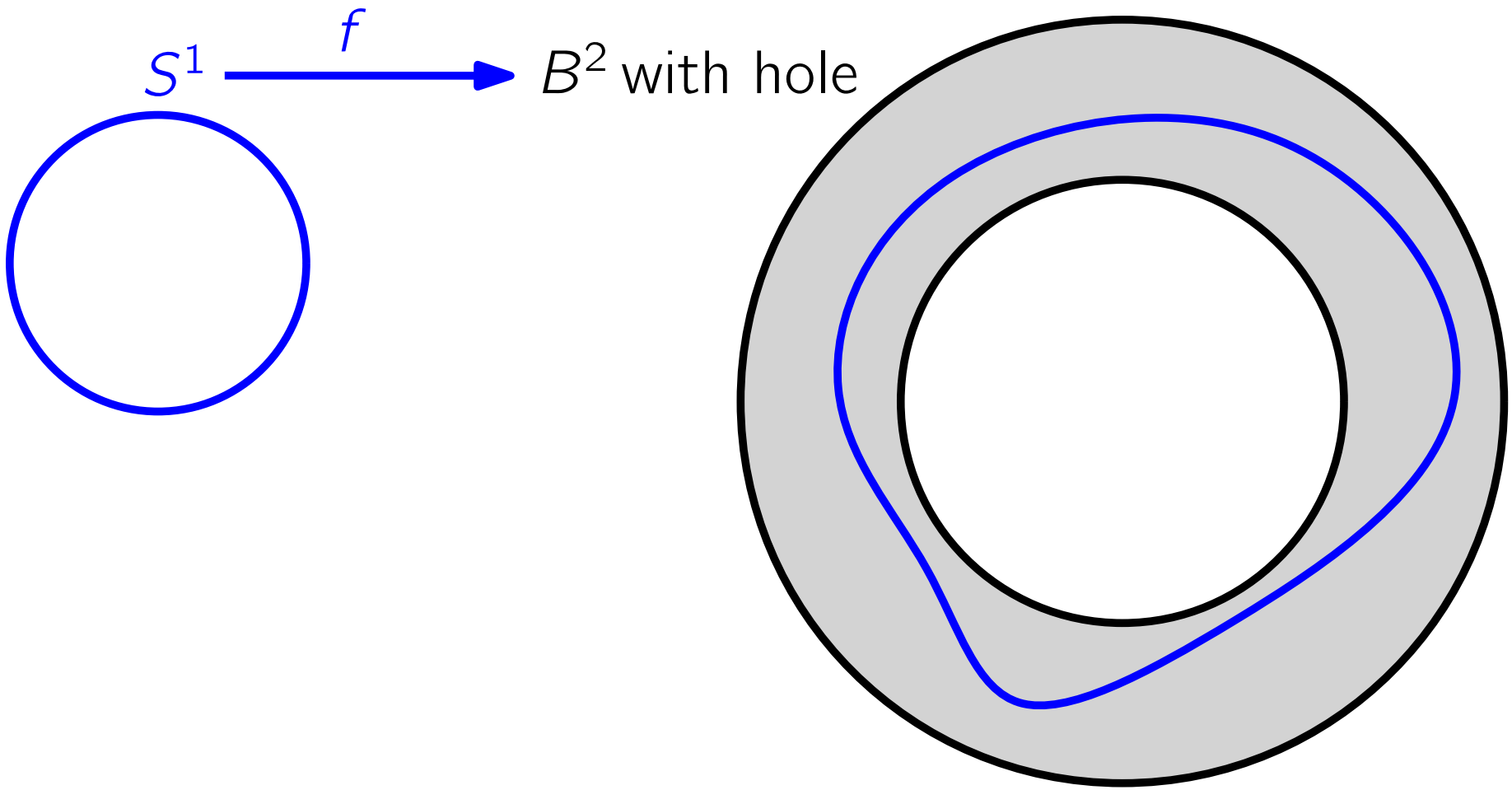

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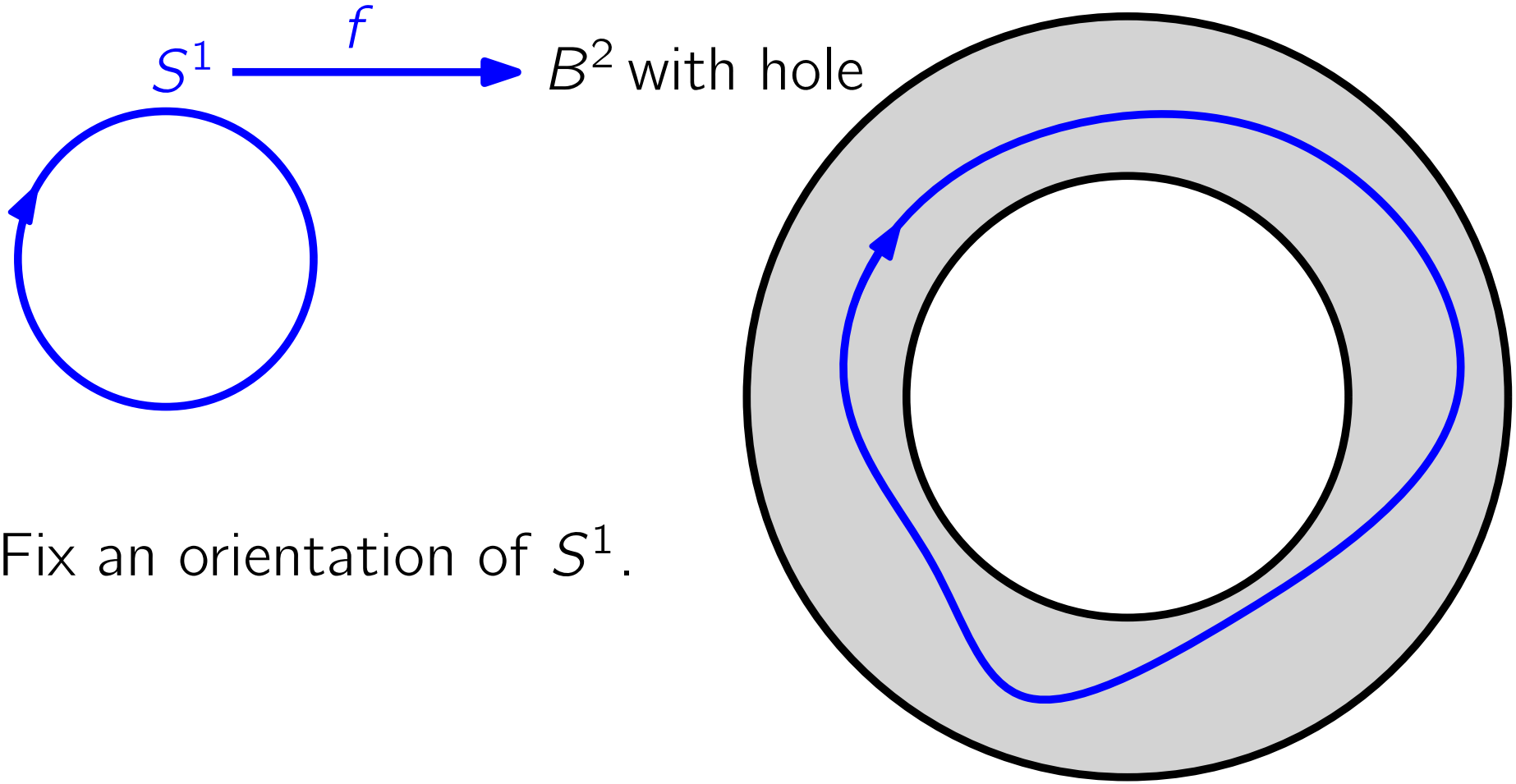

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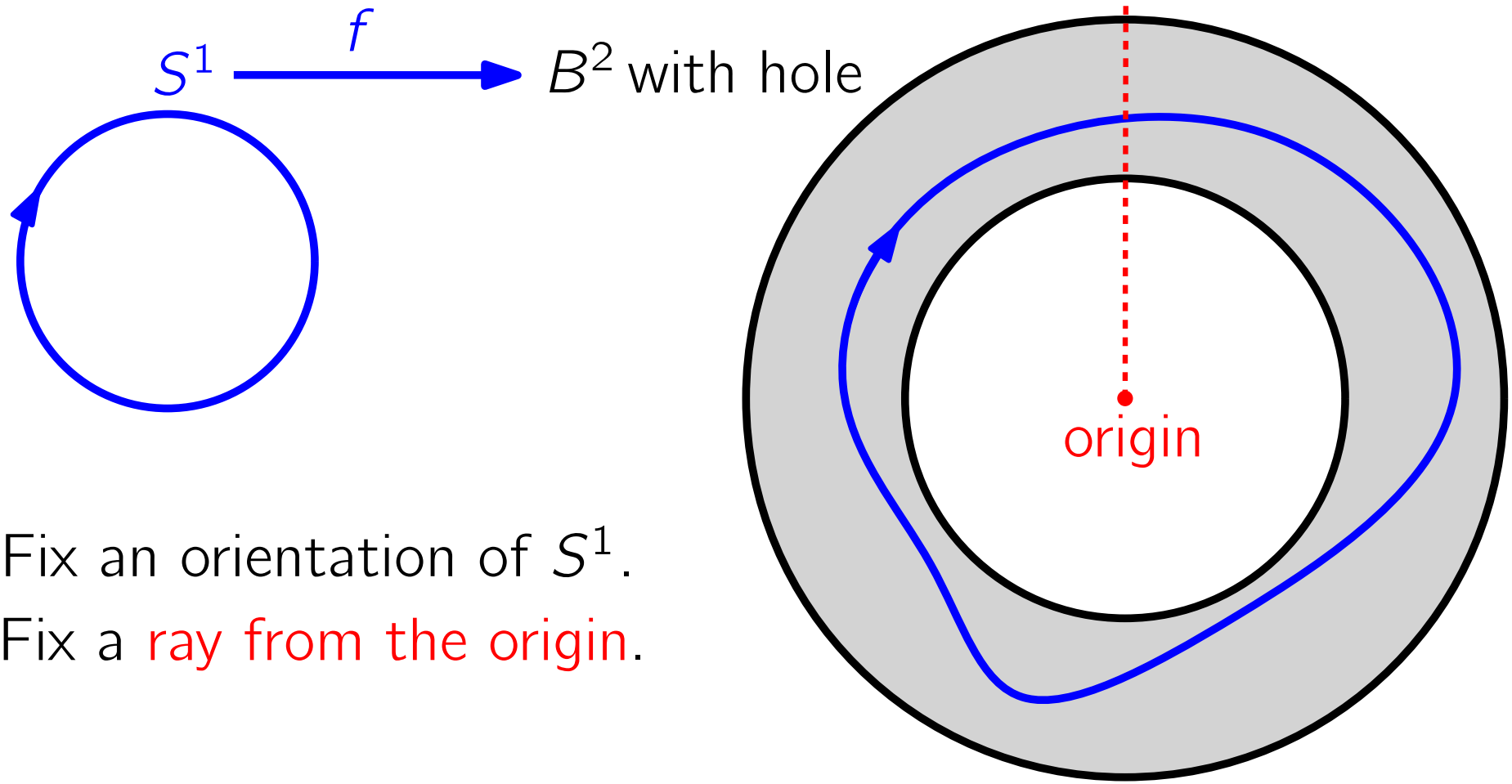

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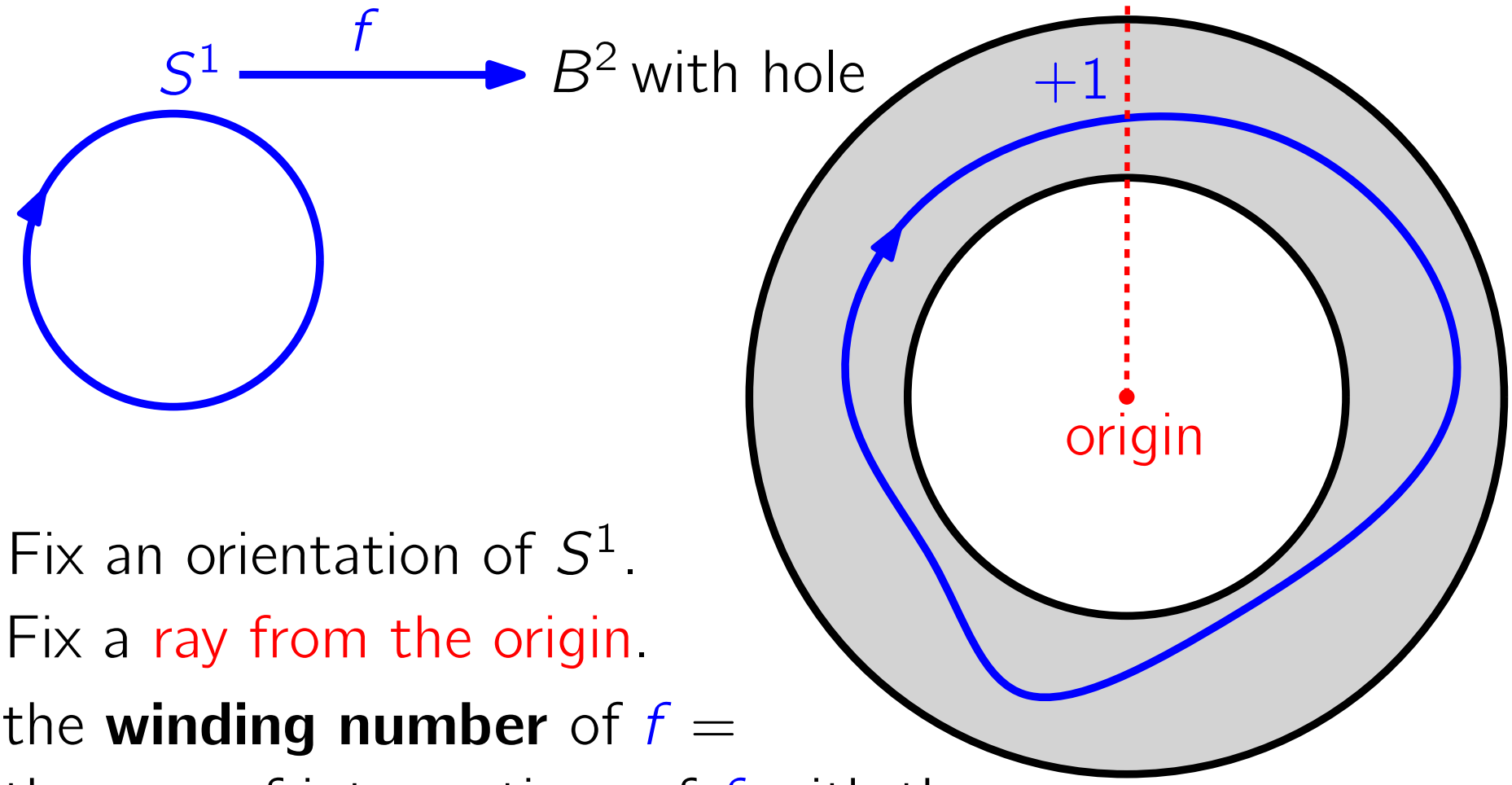

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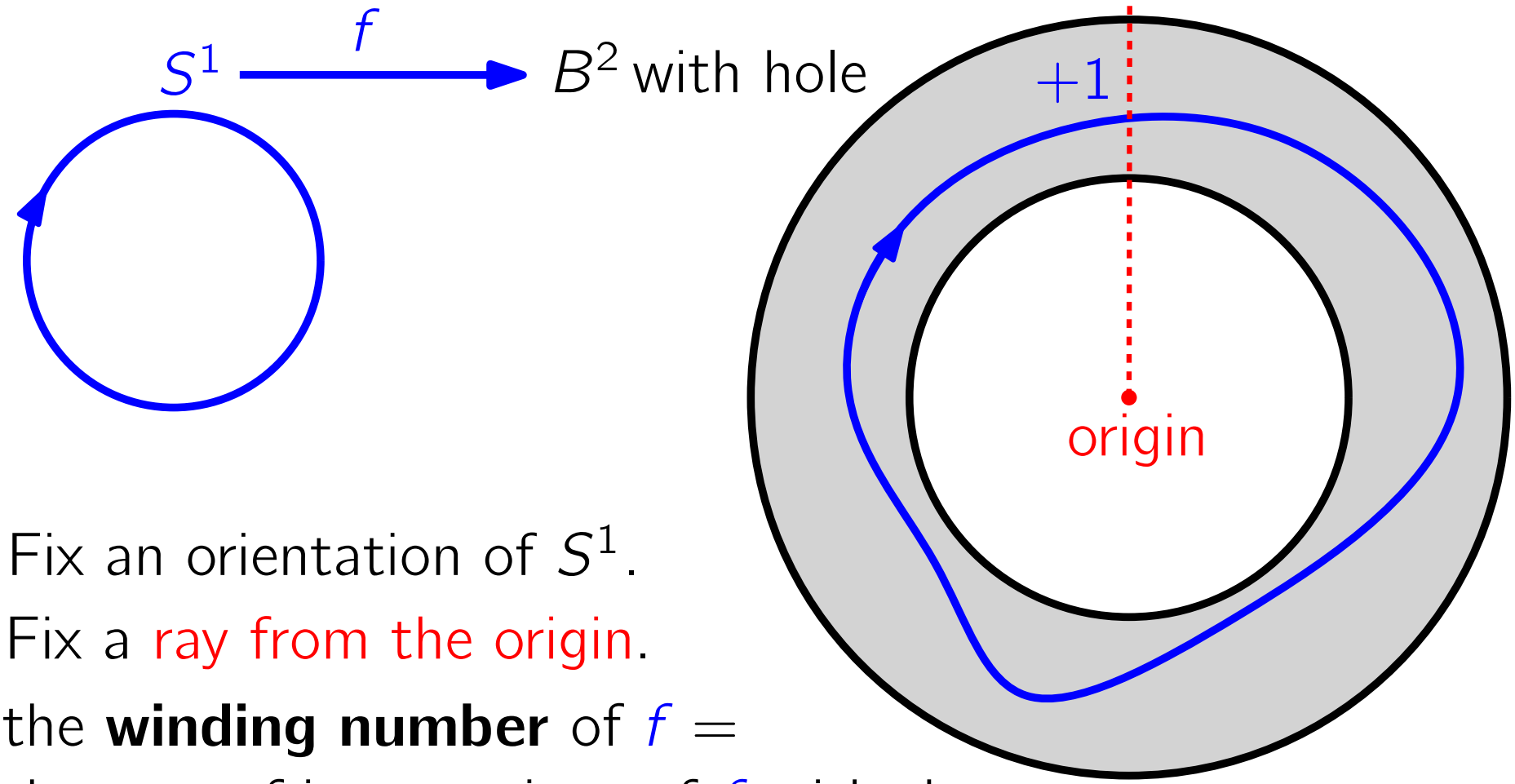

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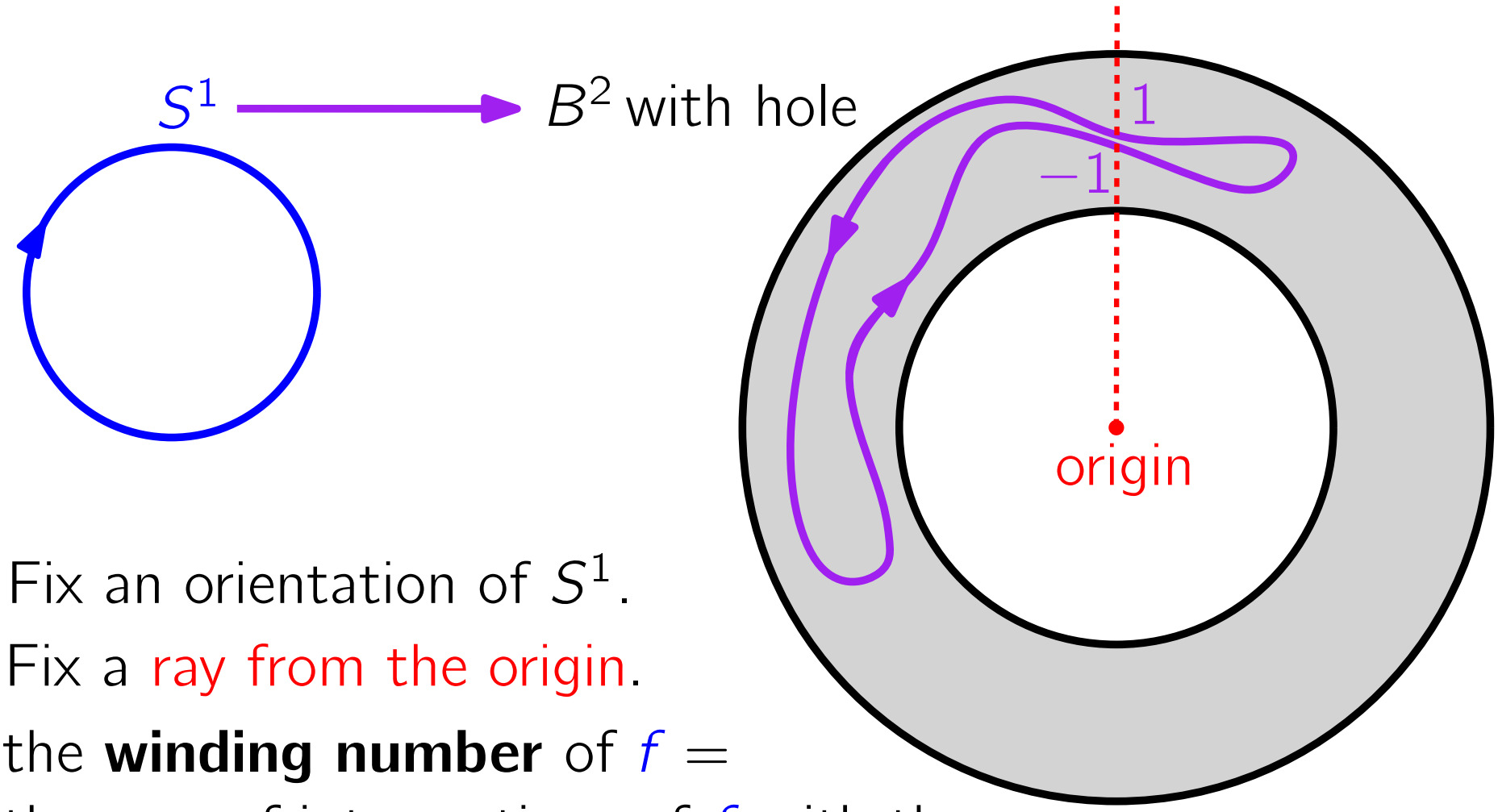

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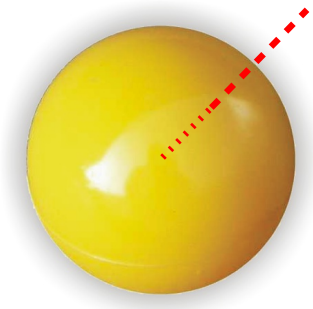

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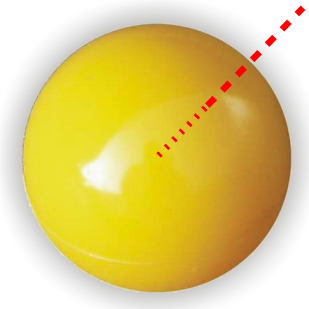

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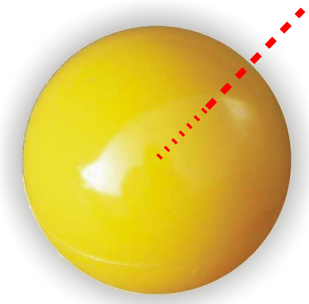

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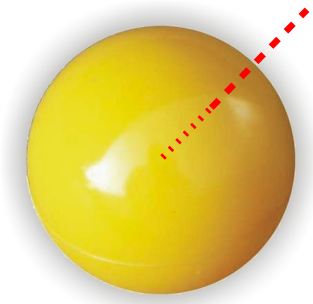

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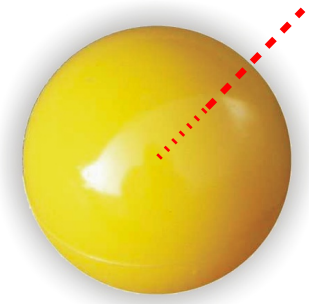

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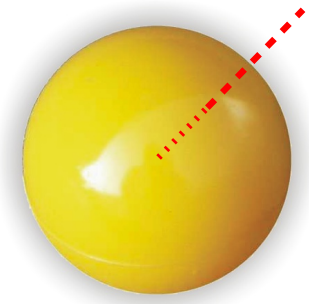

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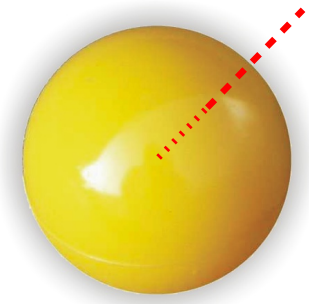

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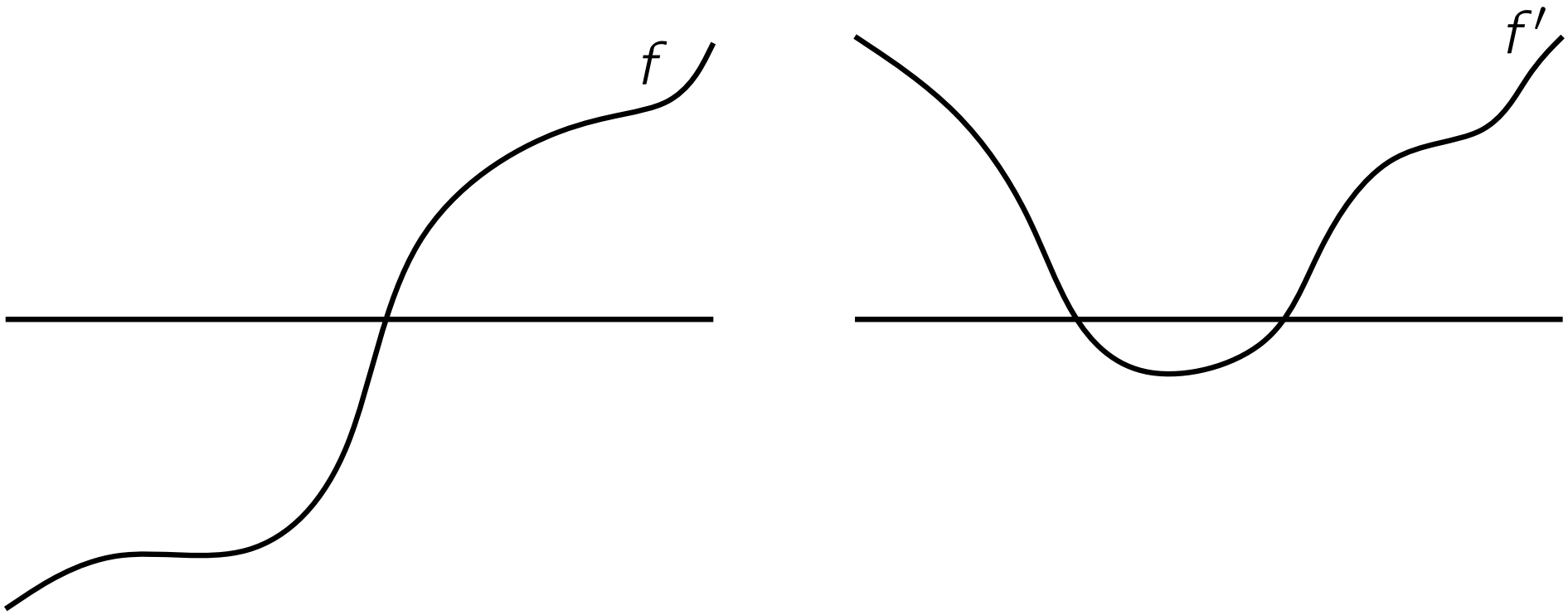

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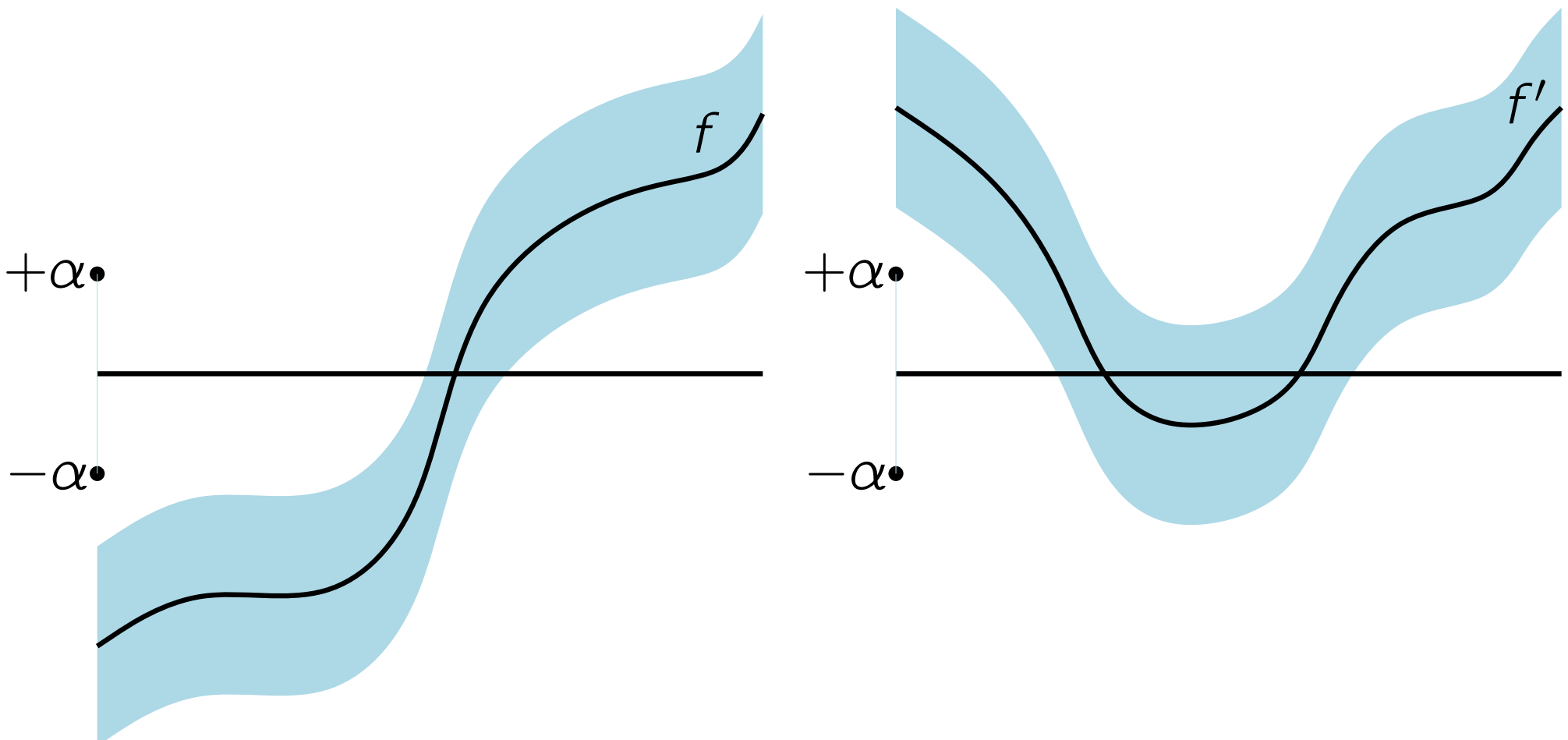

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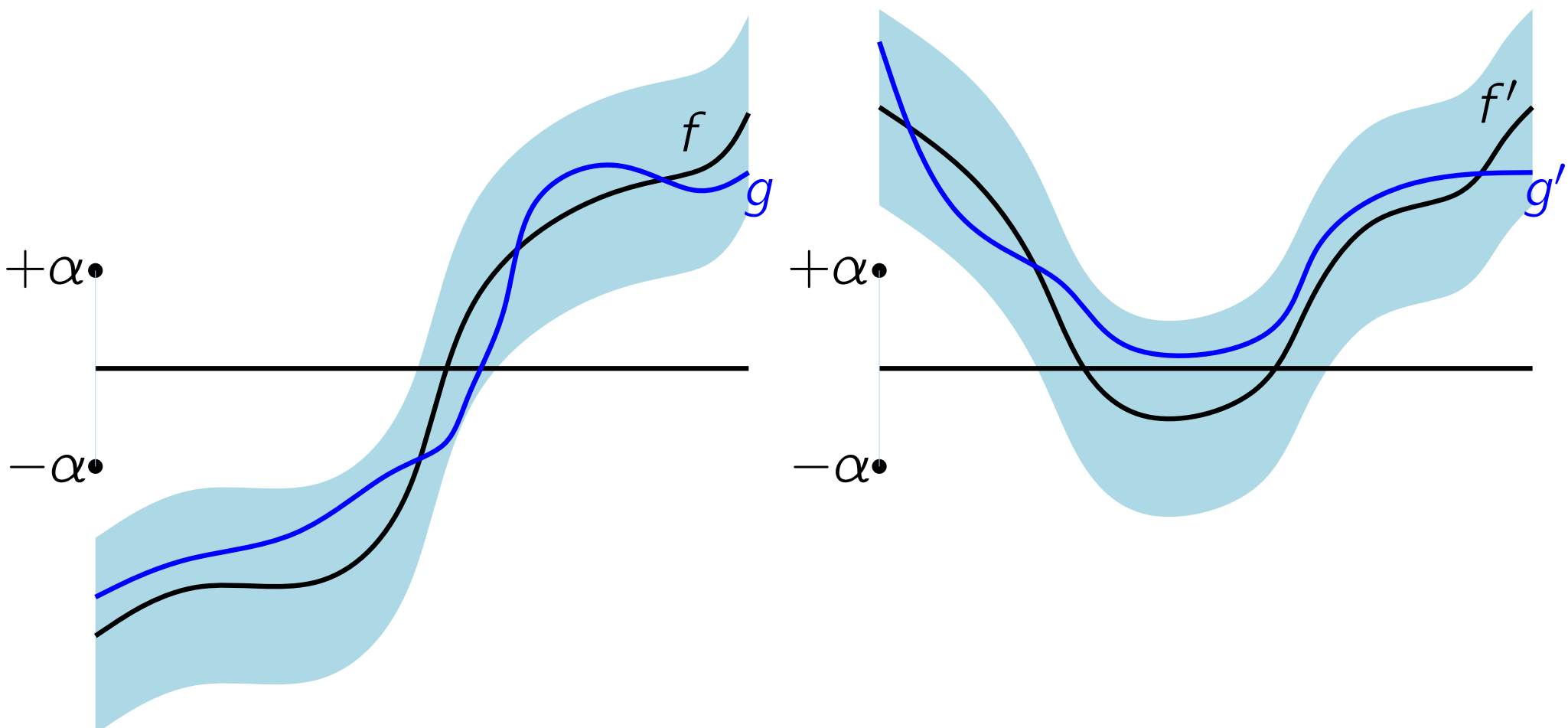

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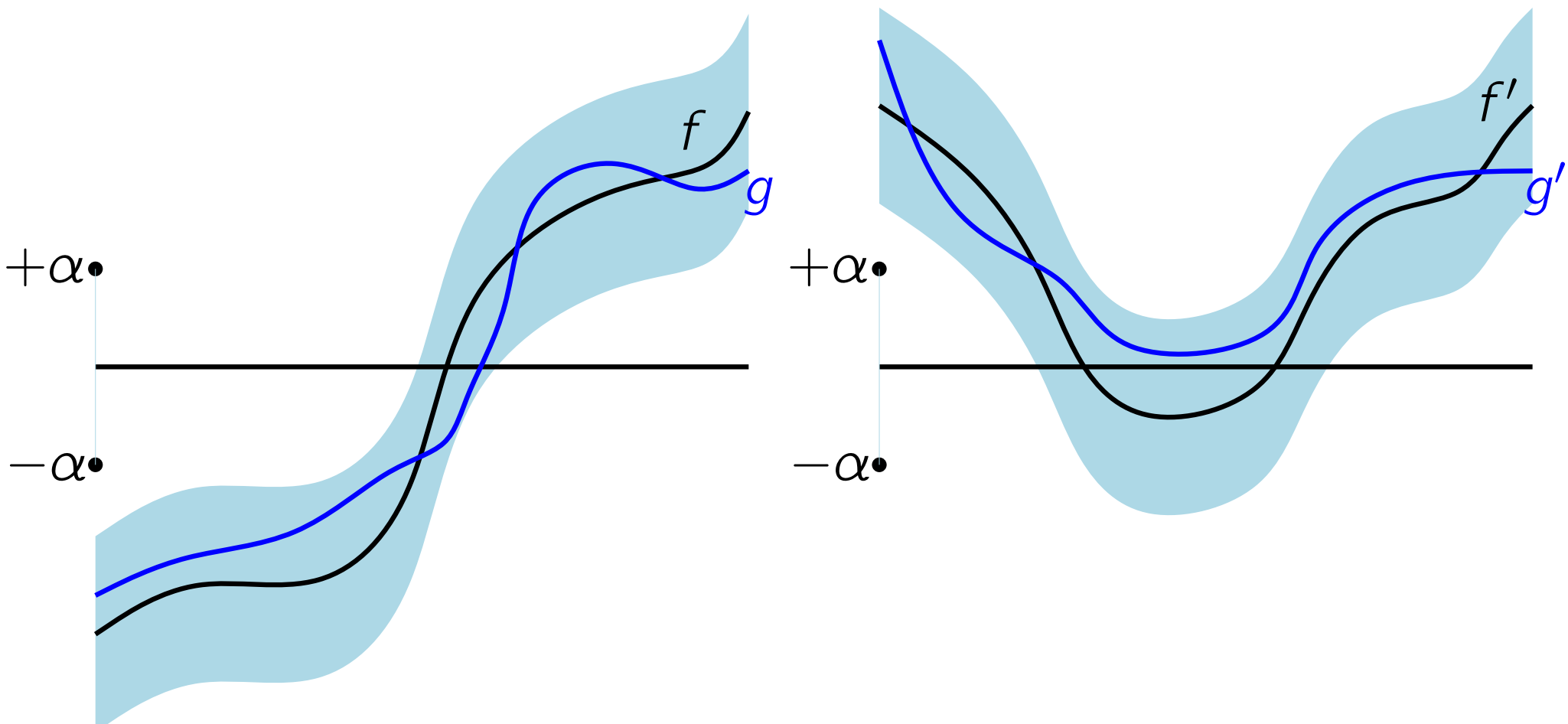

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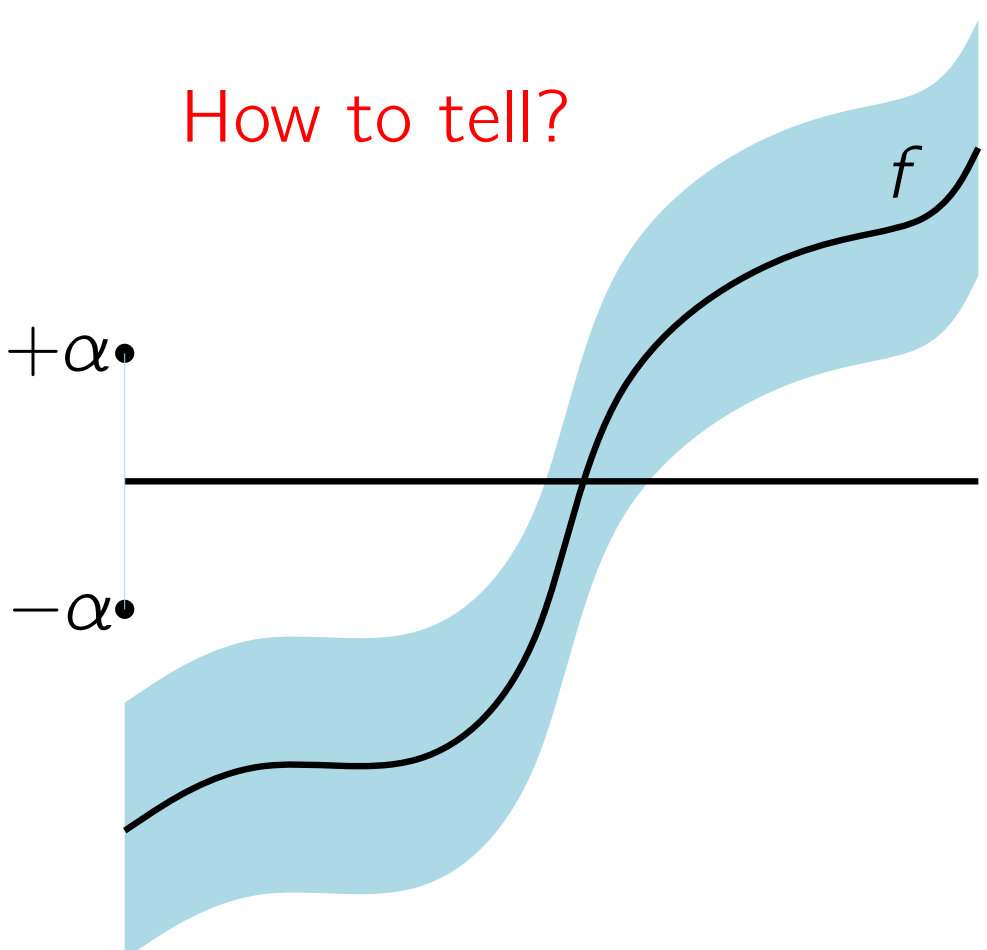

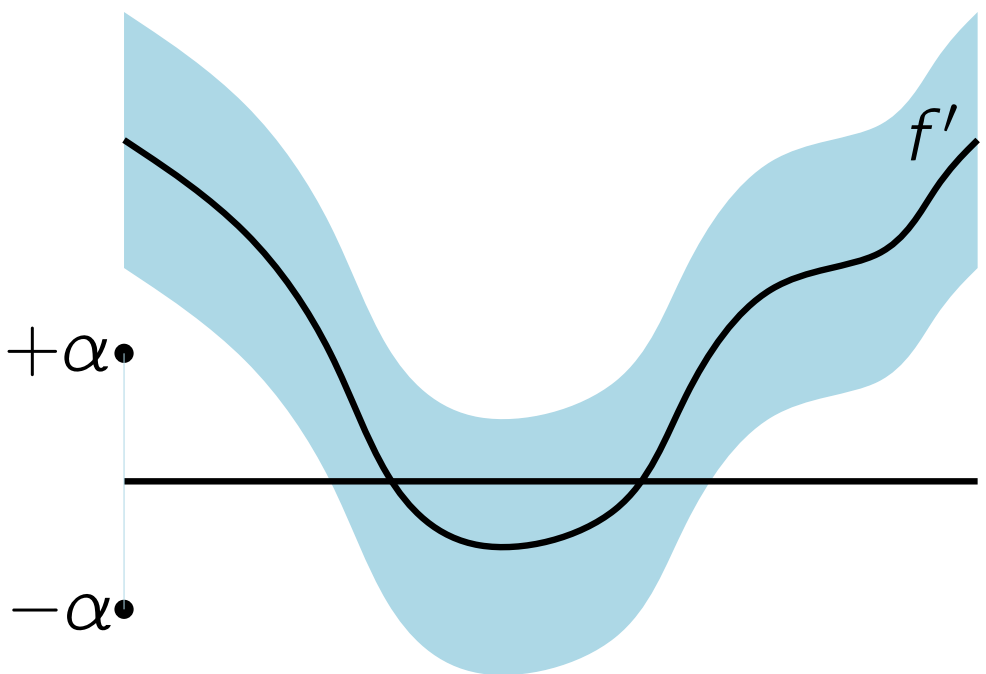

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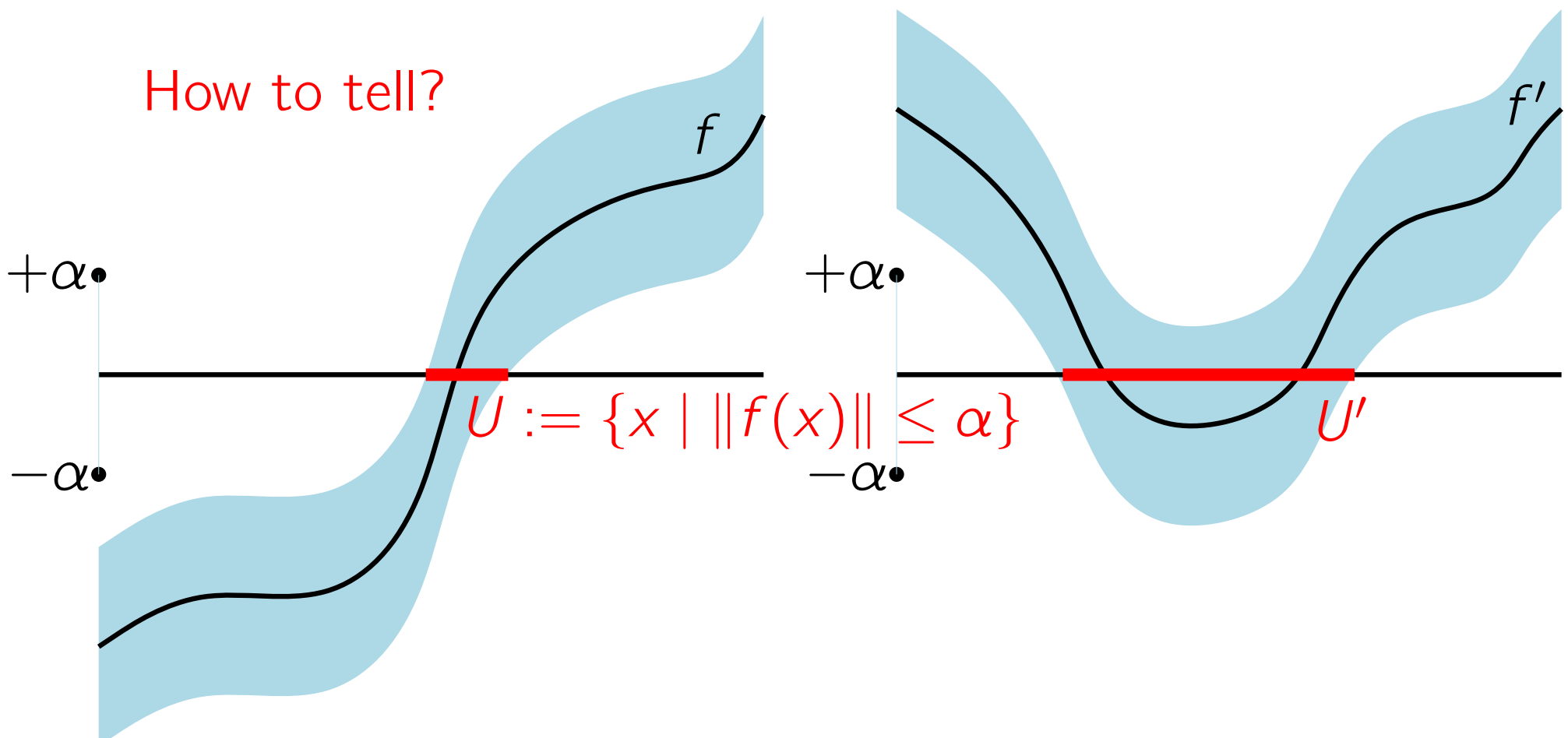

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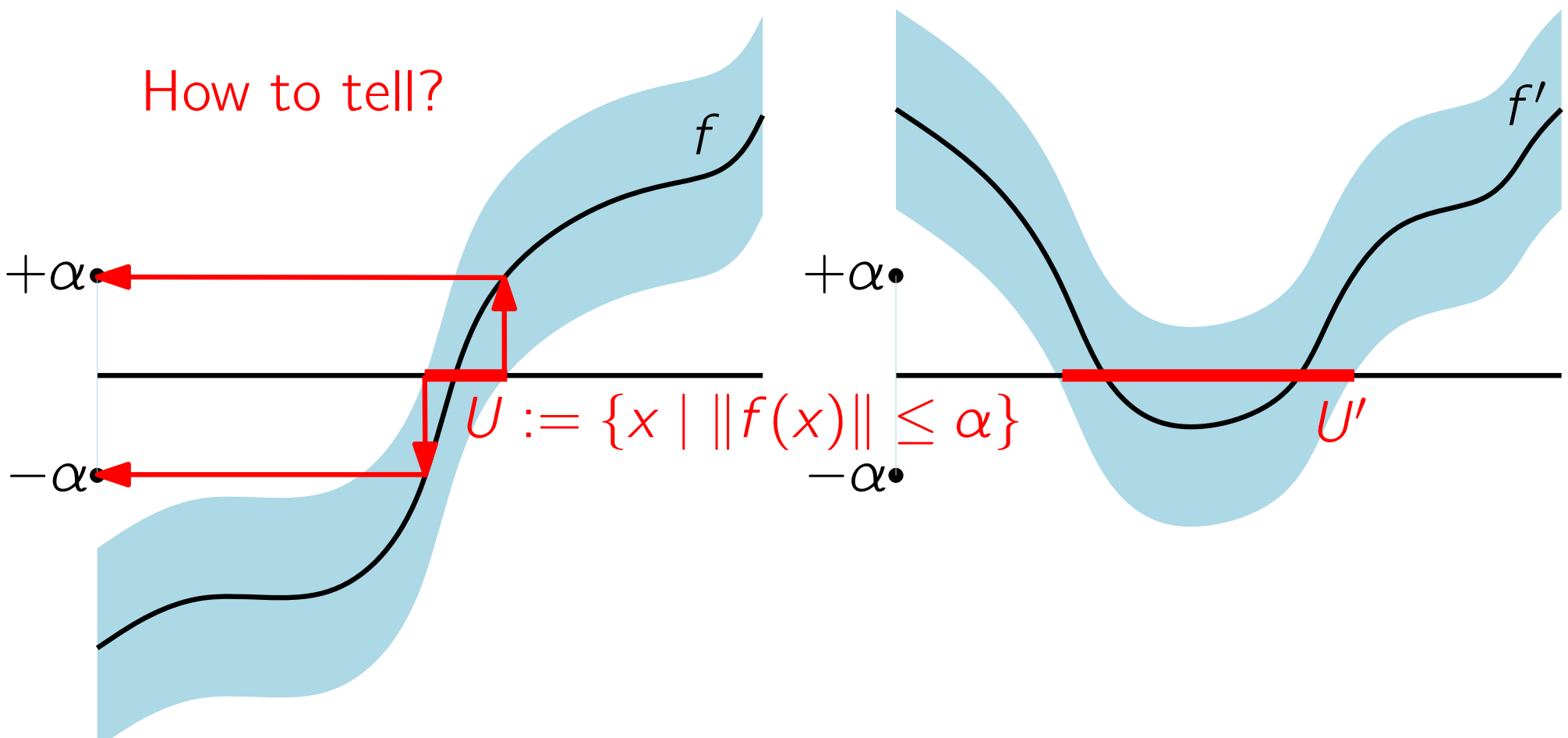

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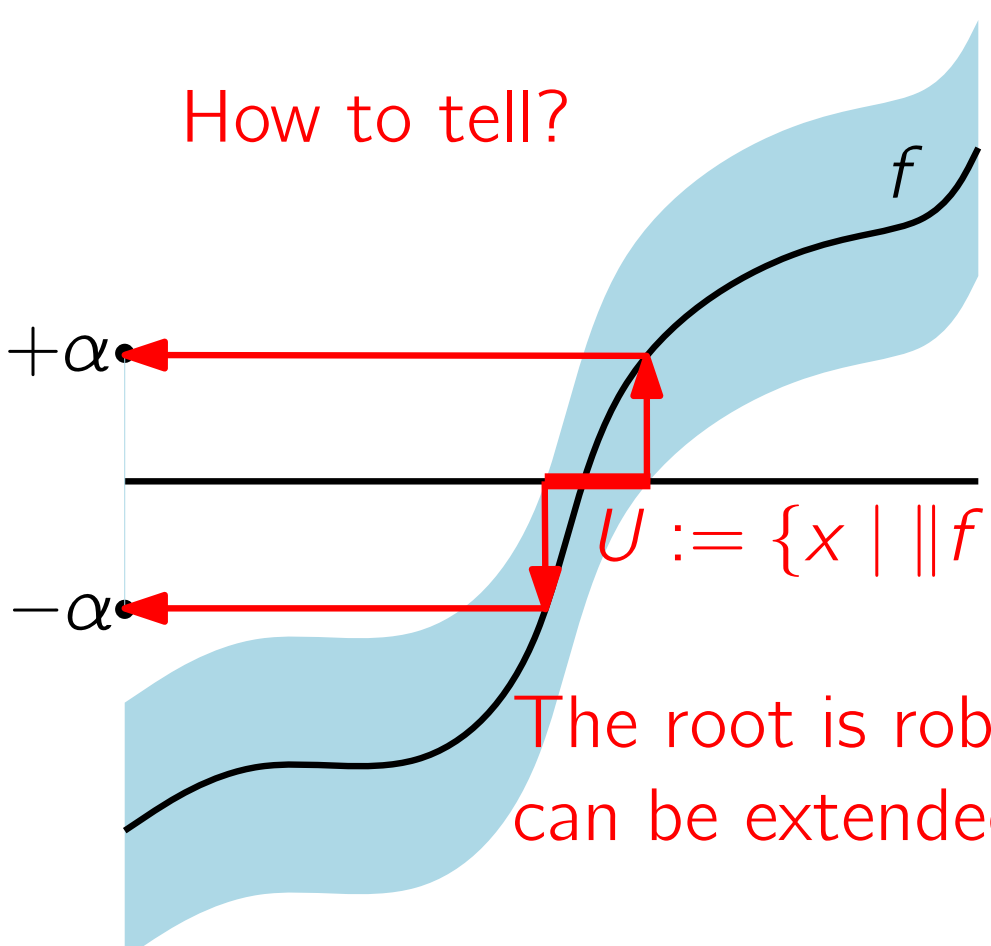

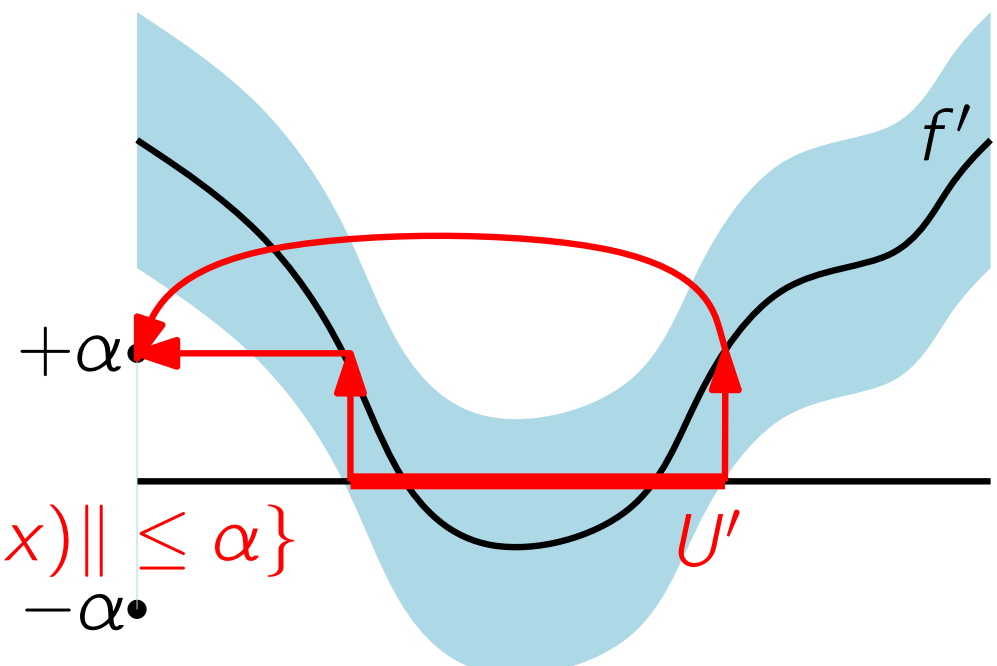

The root is robust iff the map $f : \partial U \to \{\alpha, -\alpha\}$ can be extended to all of U.

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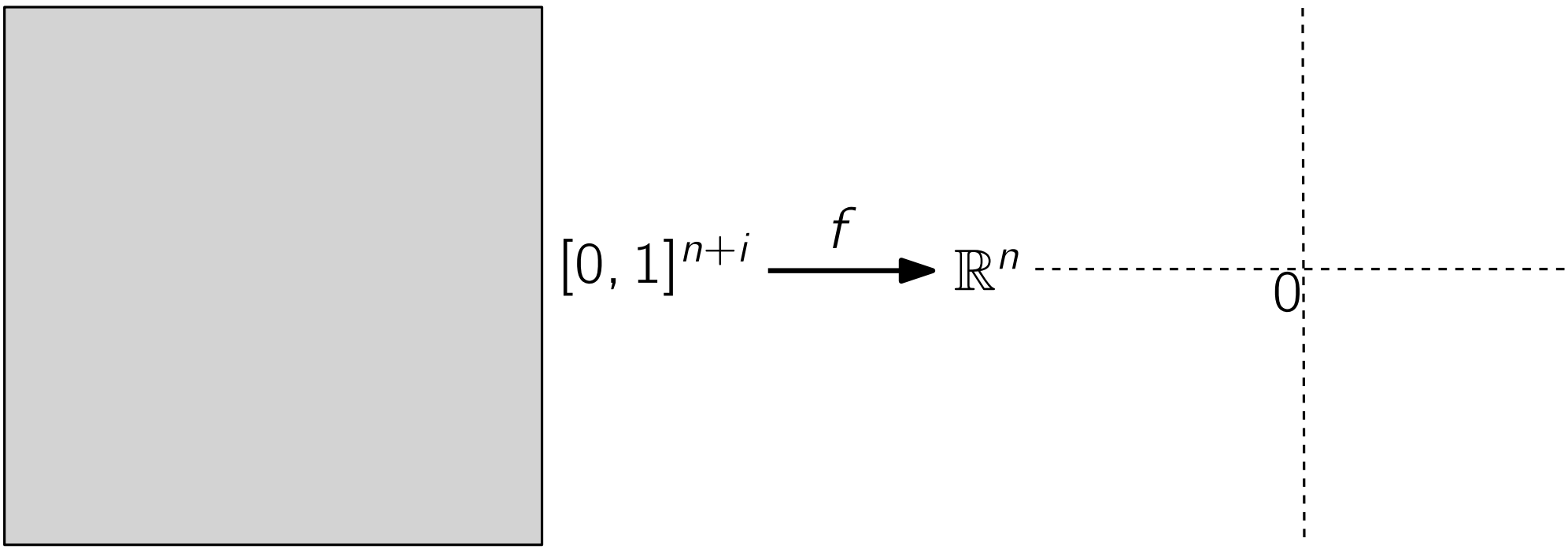

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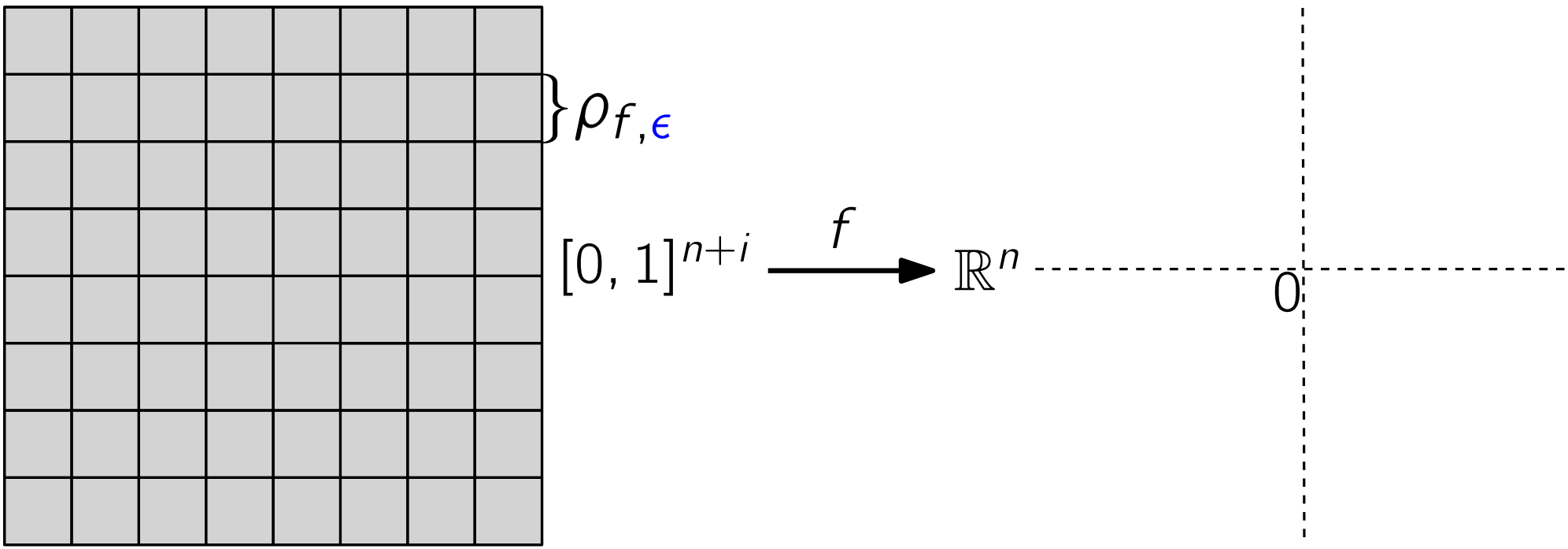

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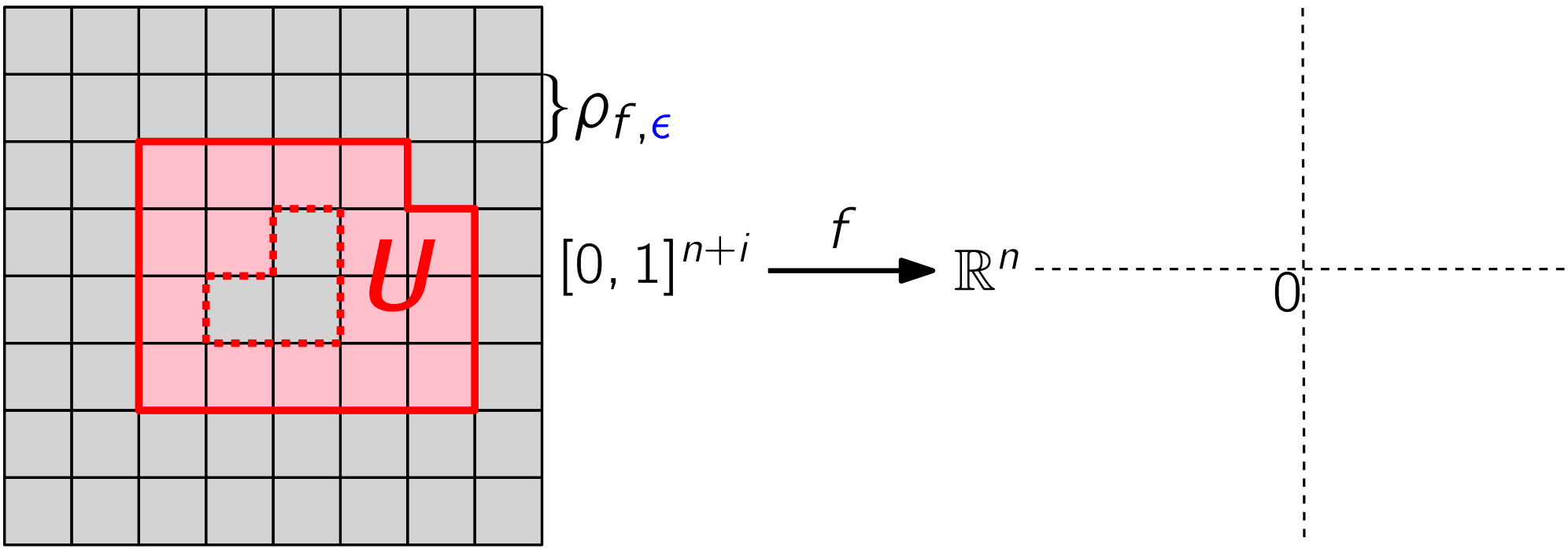

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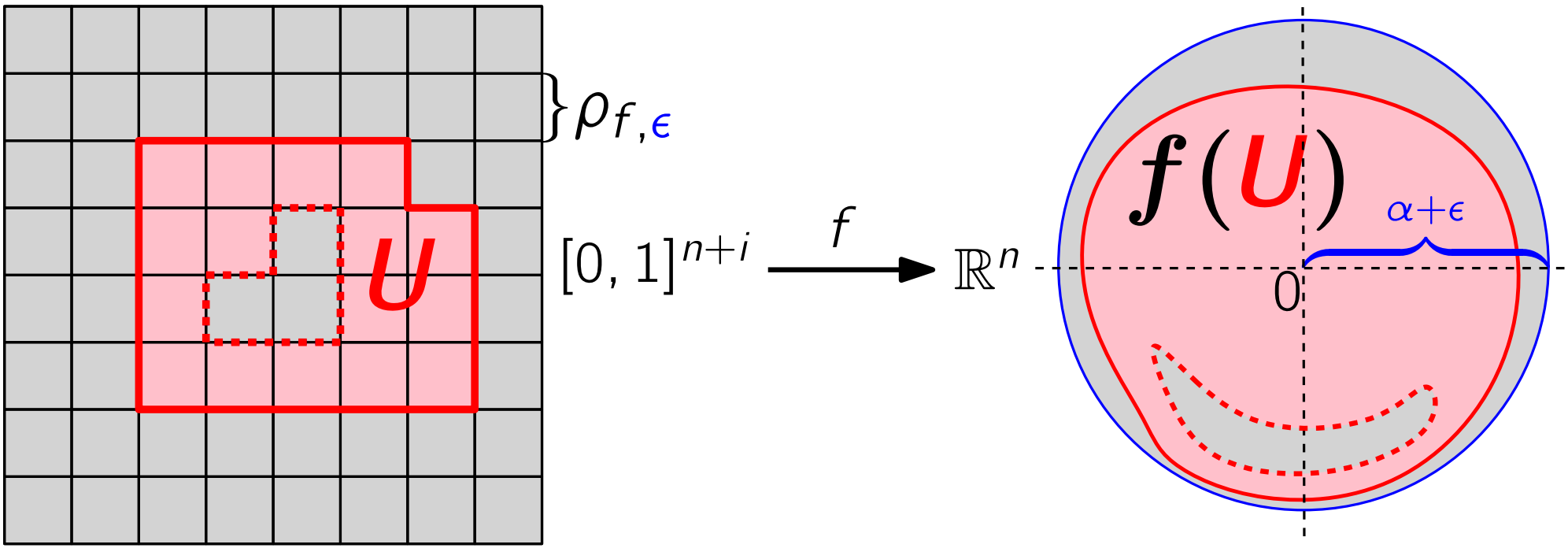

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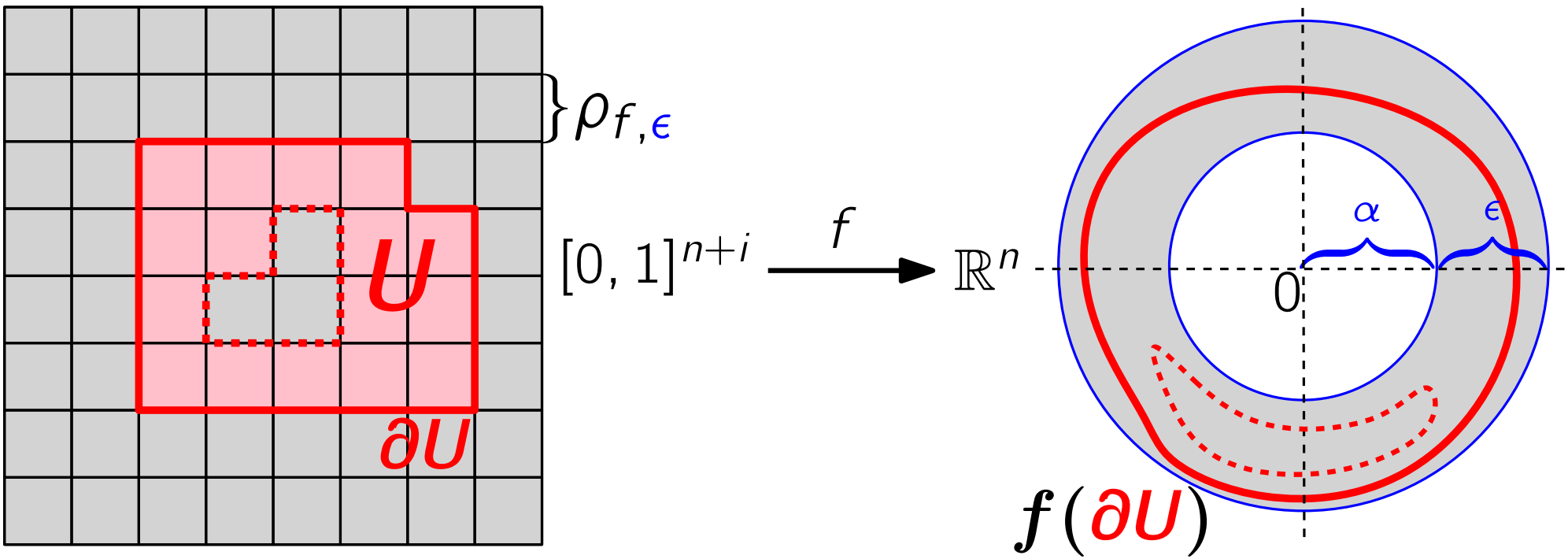

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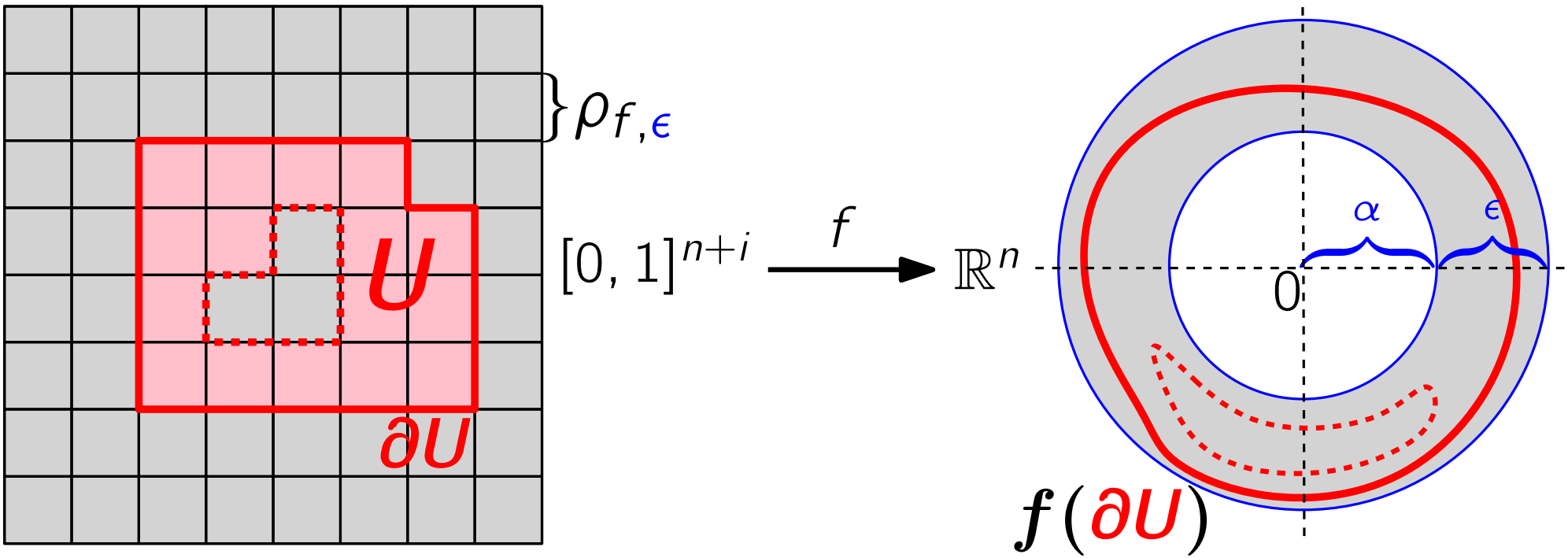

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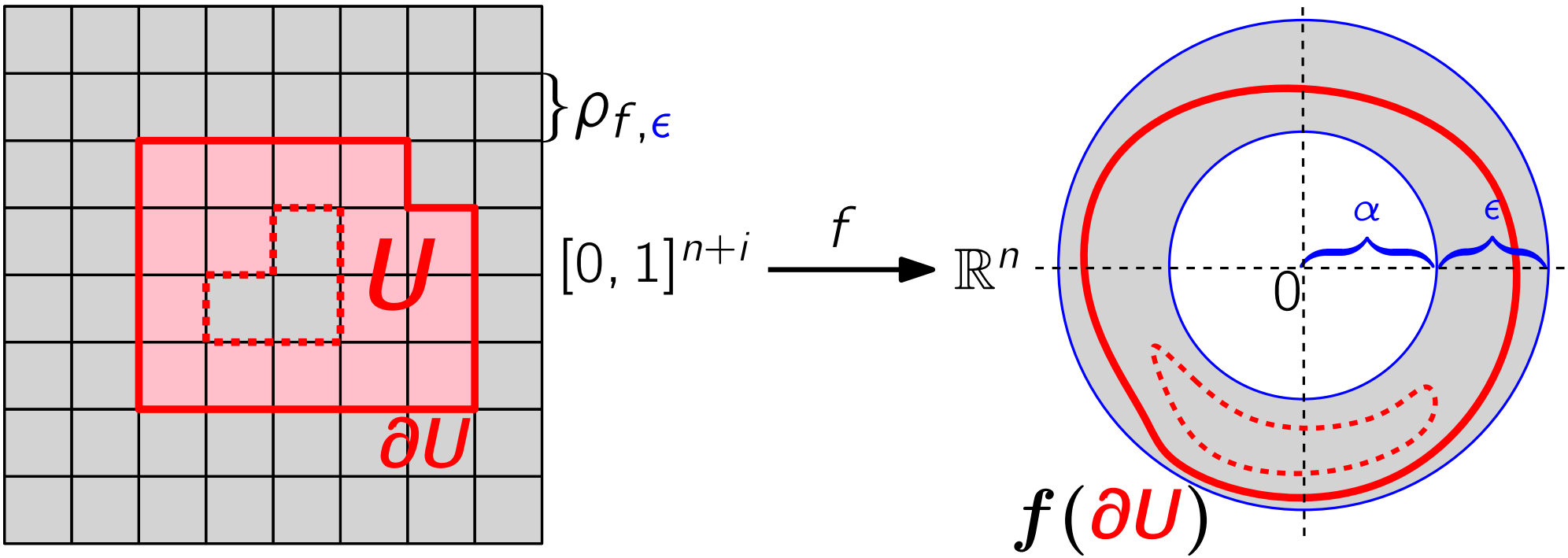

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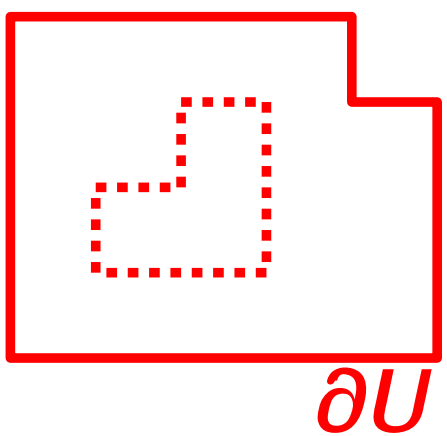

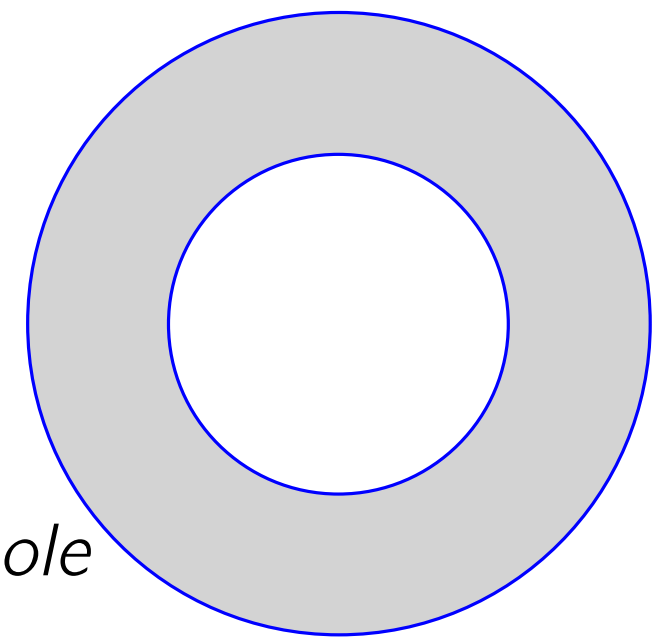

disk with hole

≀|

S^{n-1}

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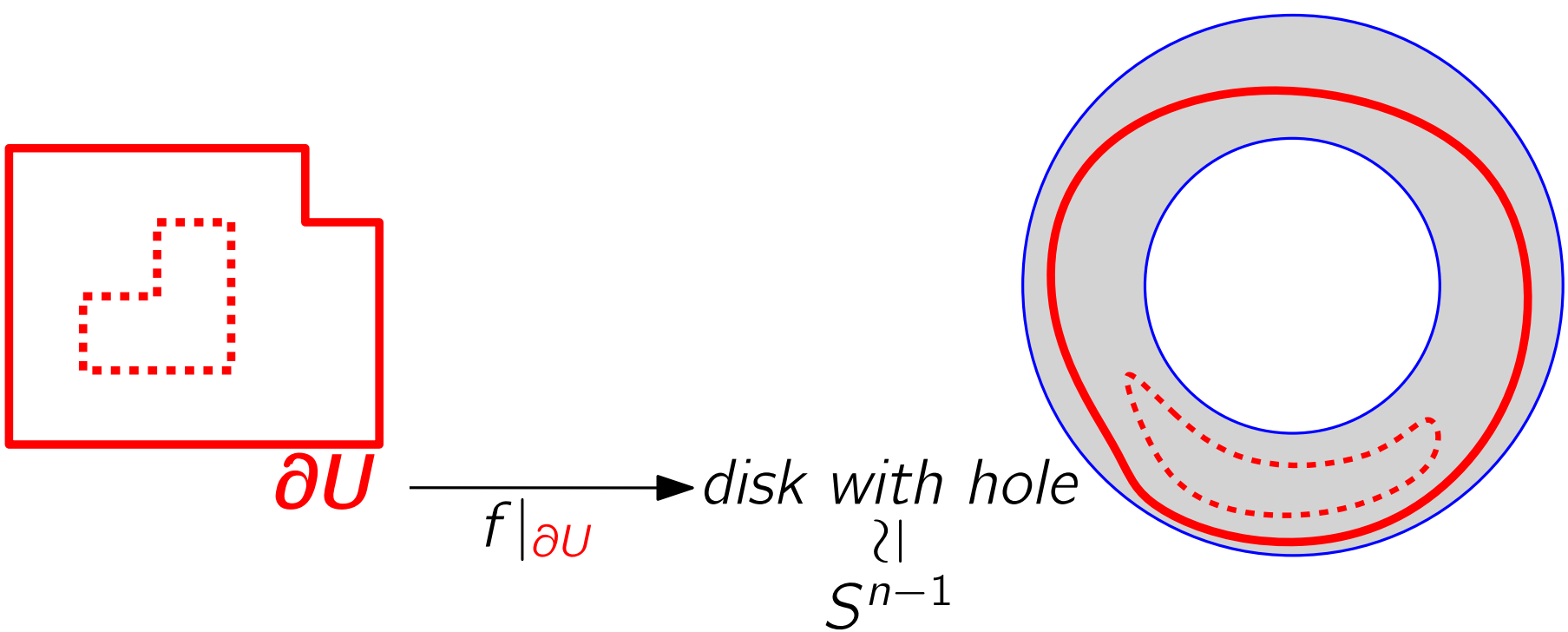

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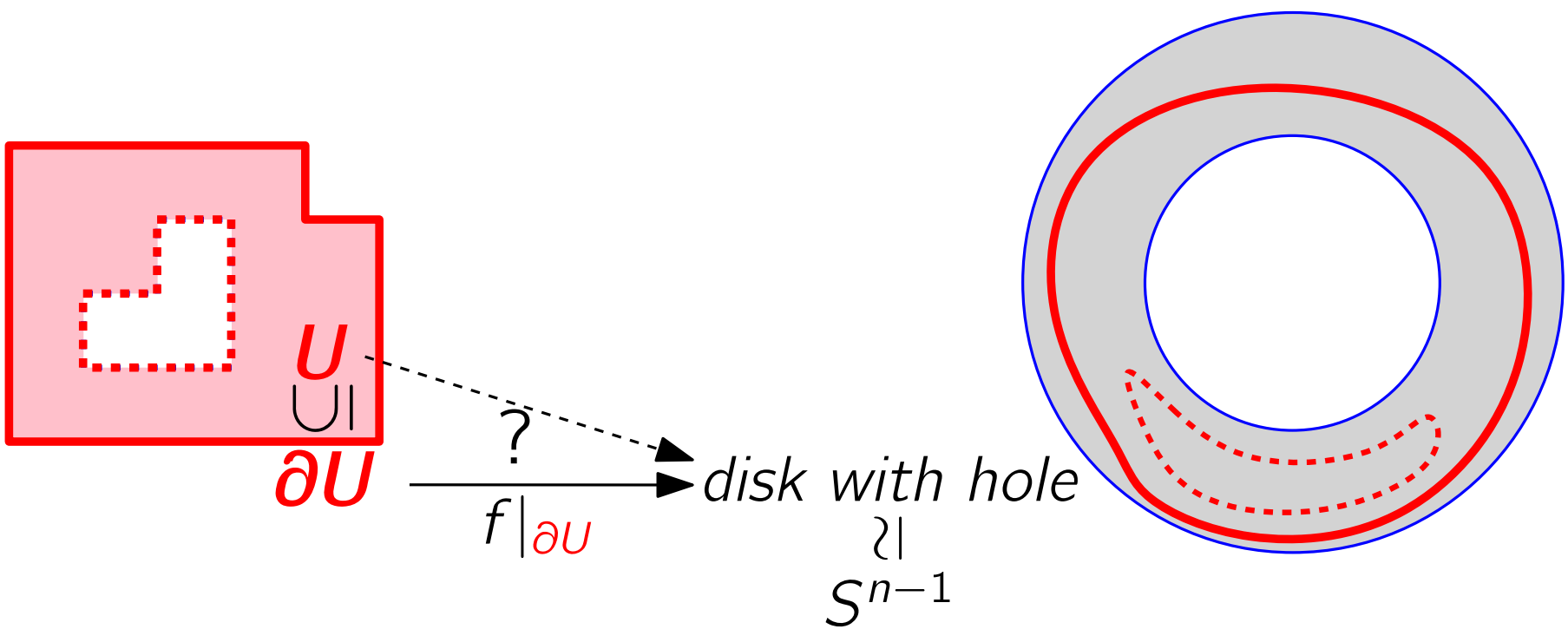

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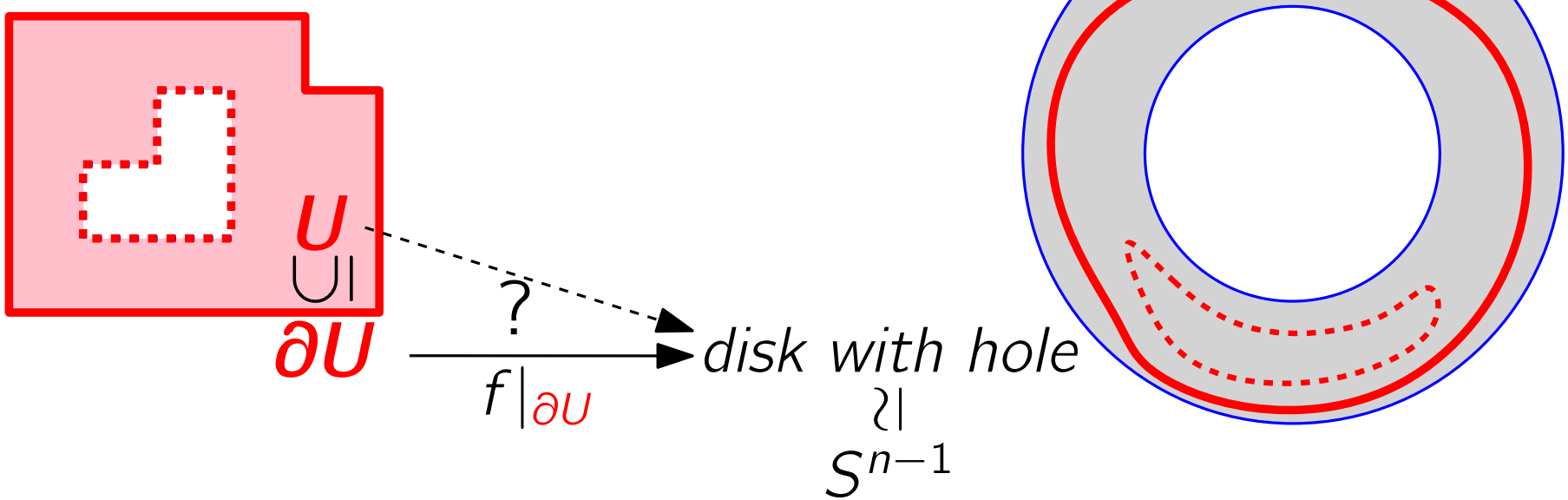

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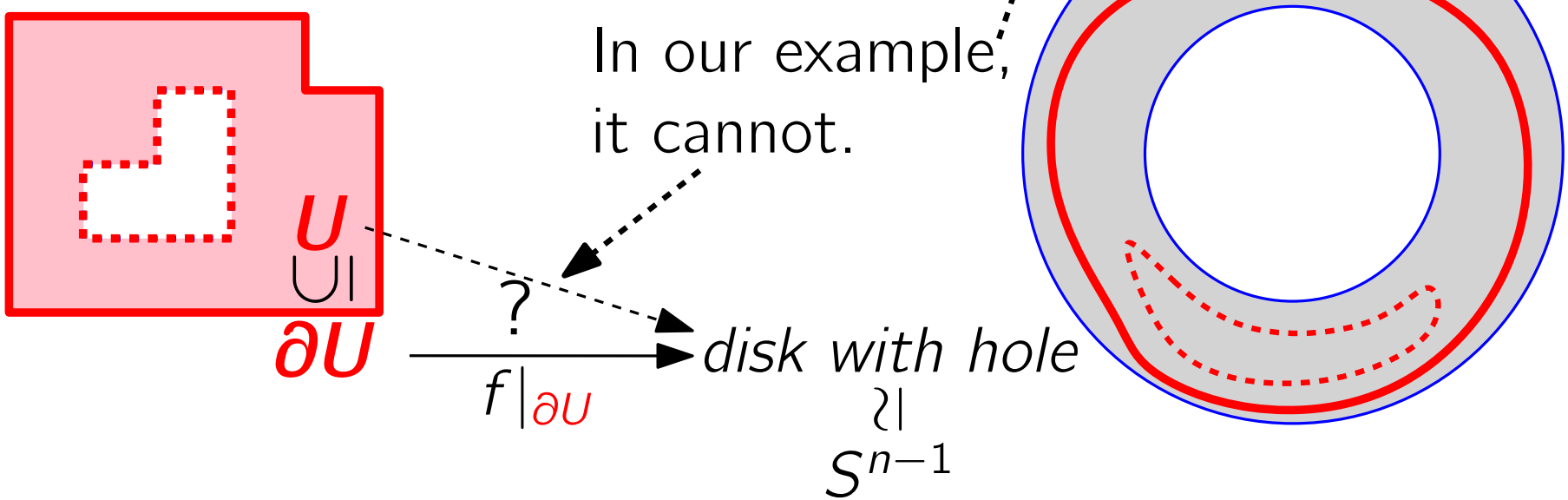

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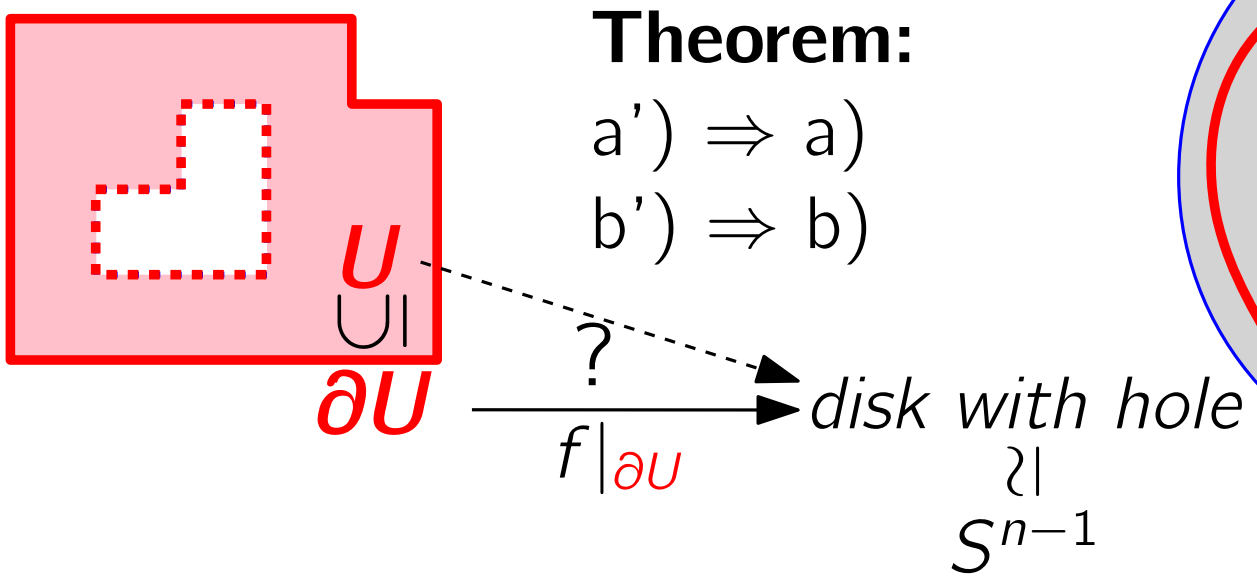

Theorem:

a') $\Rightarrow$ a)

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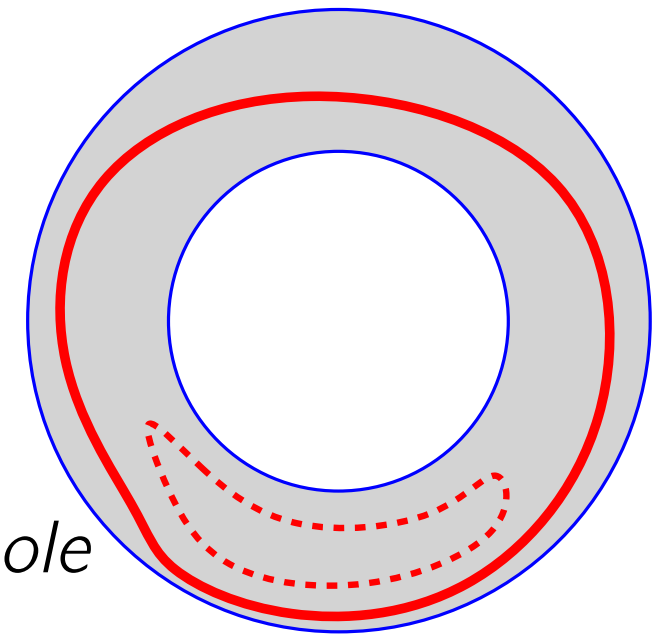

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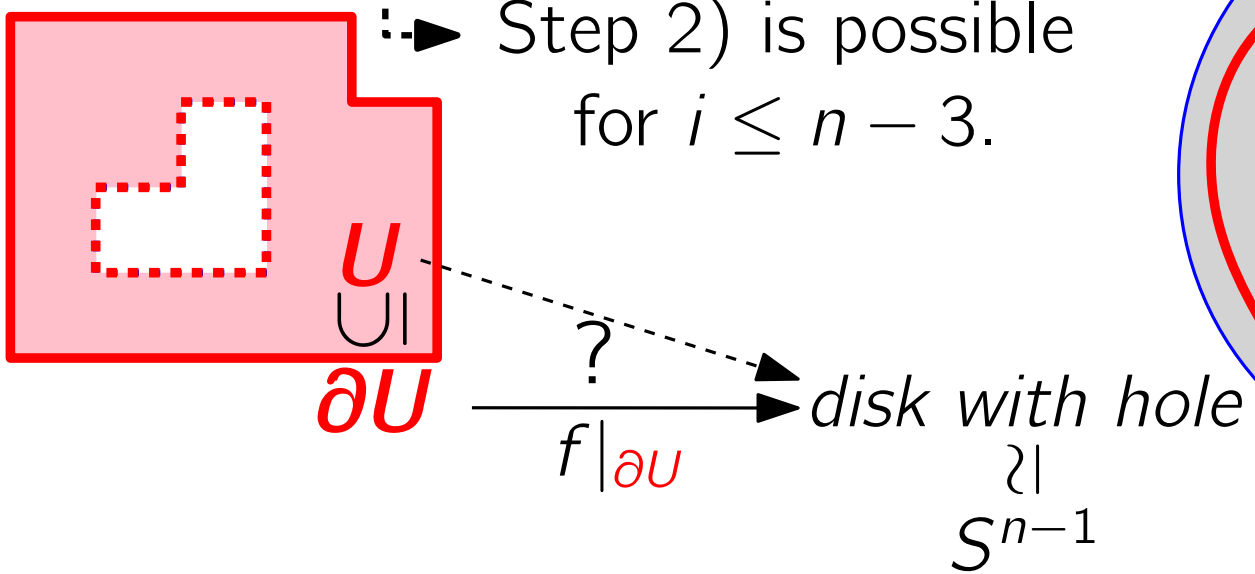
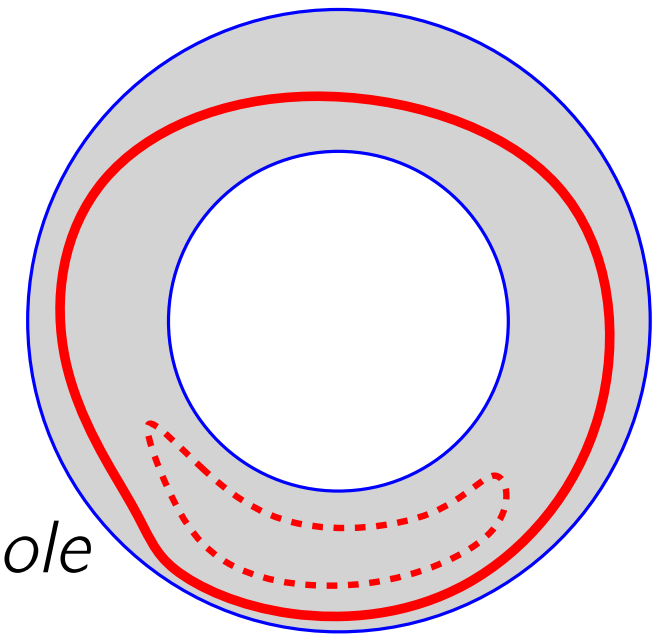
Step 2) is possible for $i \le n-3$.





$[X, Y]$ computation — the problem statement

In: simplicial complexes X and Y

Out: $[X, Y]$

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- Warning: $[X, Y]$ typically infinite

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In: simplicial complexes X and Y, Y is $(d-1)$-*connected* $([S^0, Y] = \ldots = [S^{d-1}, Y] = 0)$, $\dim X \leq 2d-2$

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- $Y = S^d$ is a fundamental example.

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- Warning: $[X, Y]$ typically infinite, BUT in the **stable range** $[X, Y]$ has an Abelian group structure

Main Theorem. The problem above is computable.

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Extendability — a related problem

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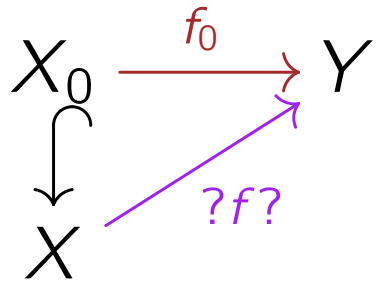

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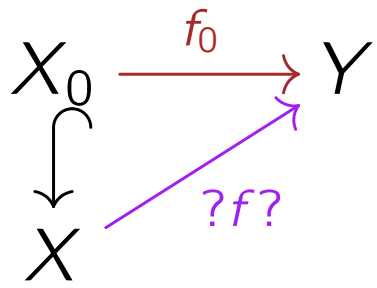

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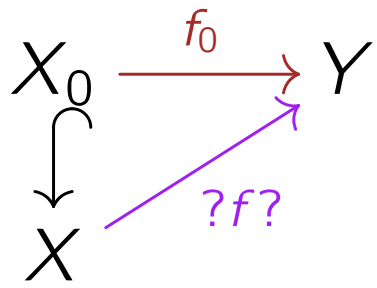

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Corollary. The extendability problem is decidable.

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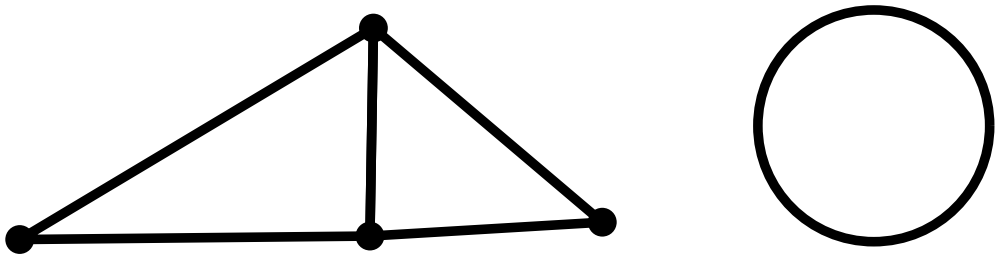

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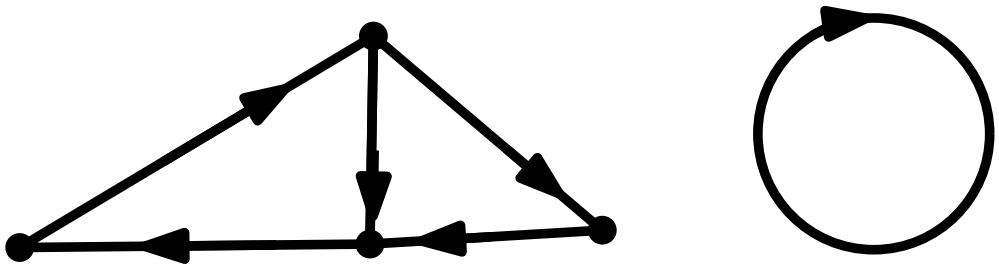

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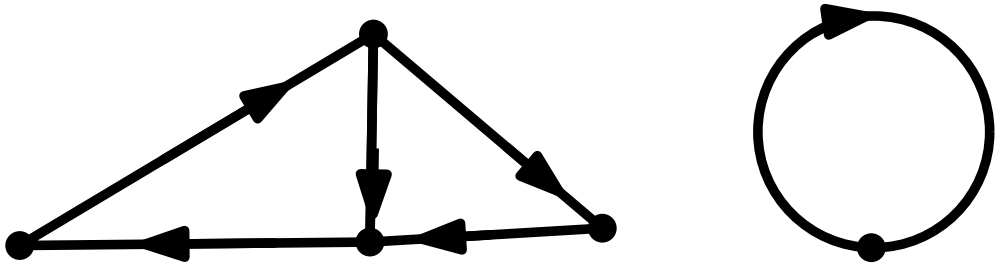

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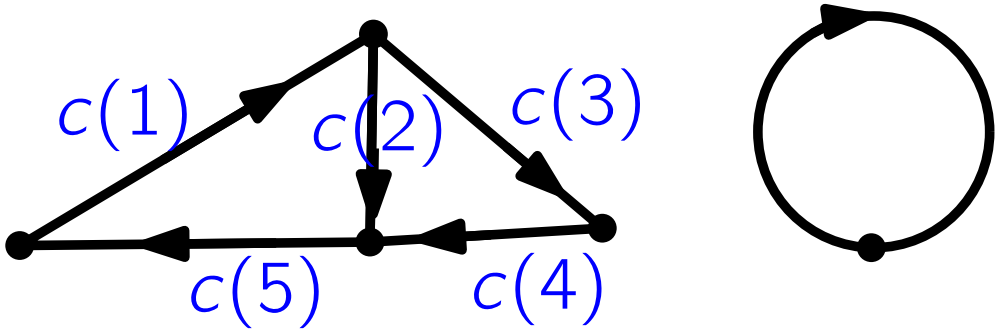

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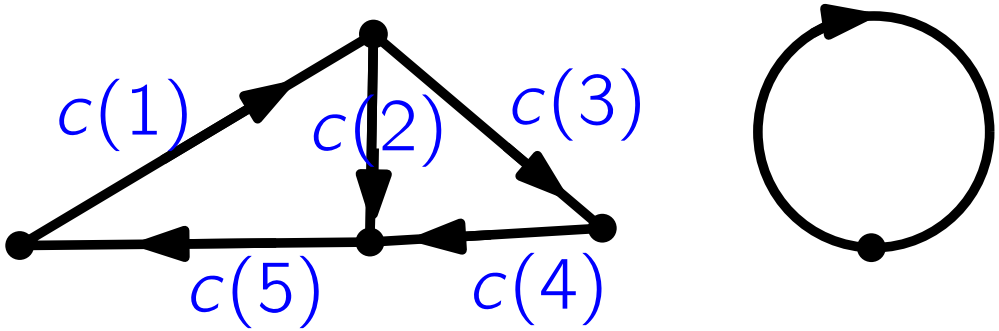

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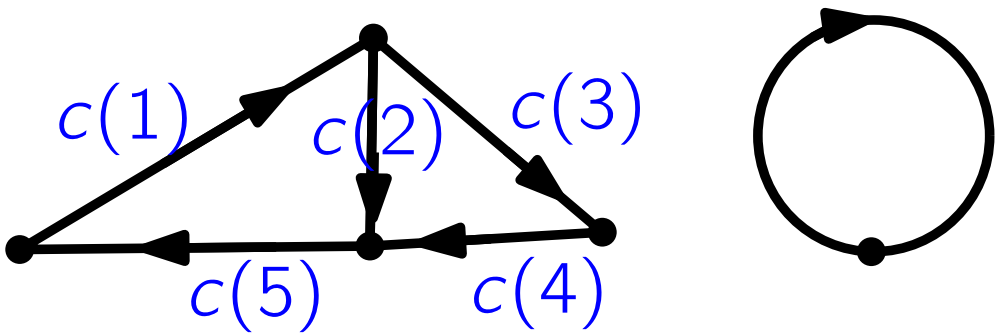

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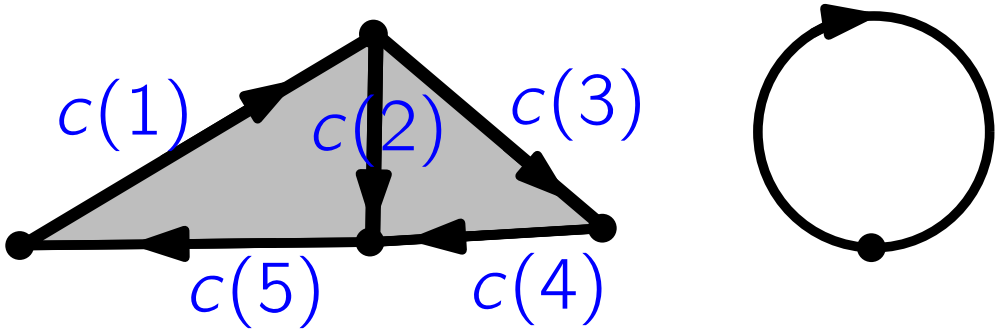

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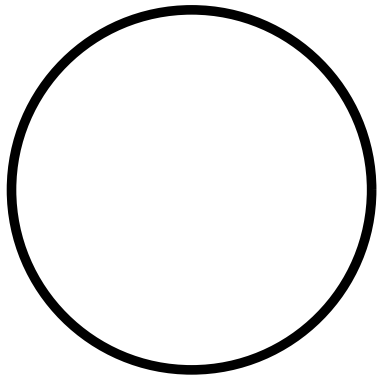

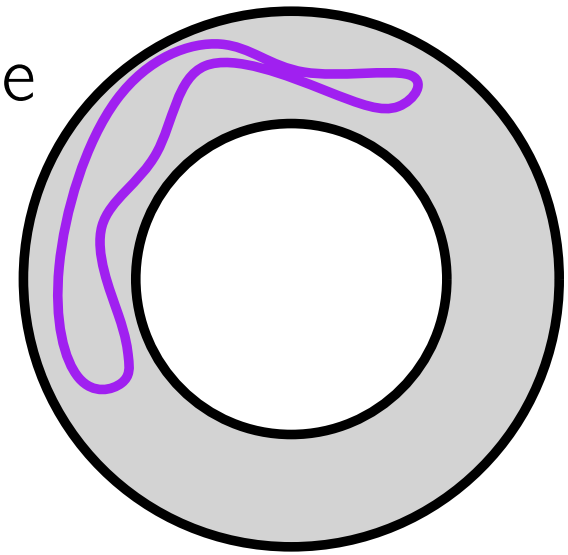

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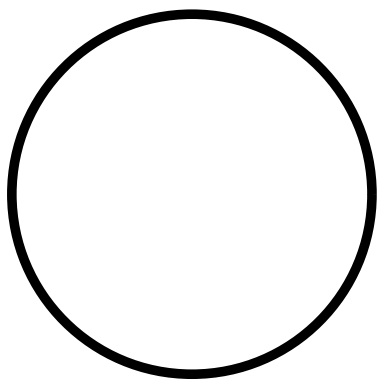

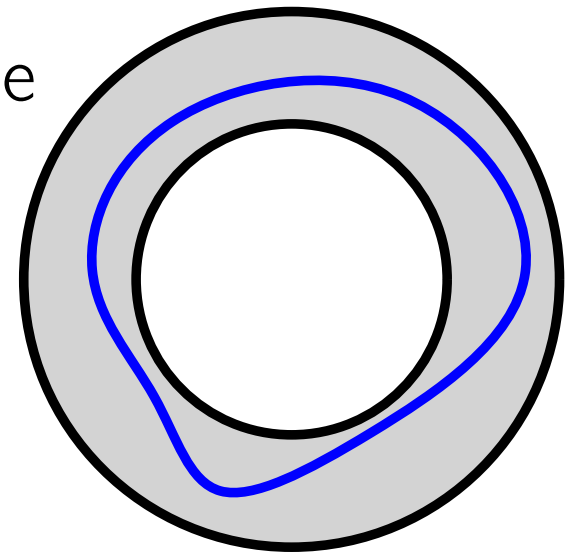

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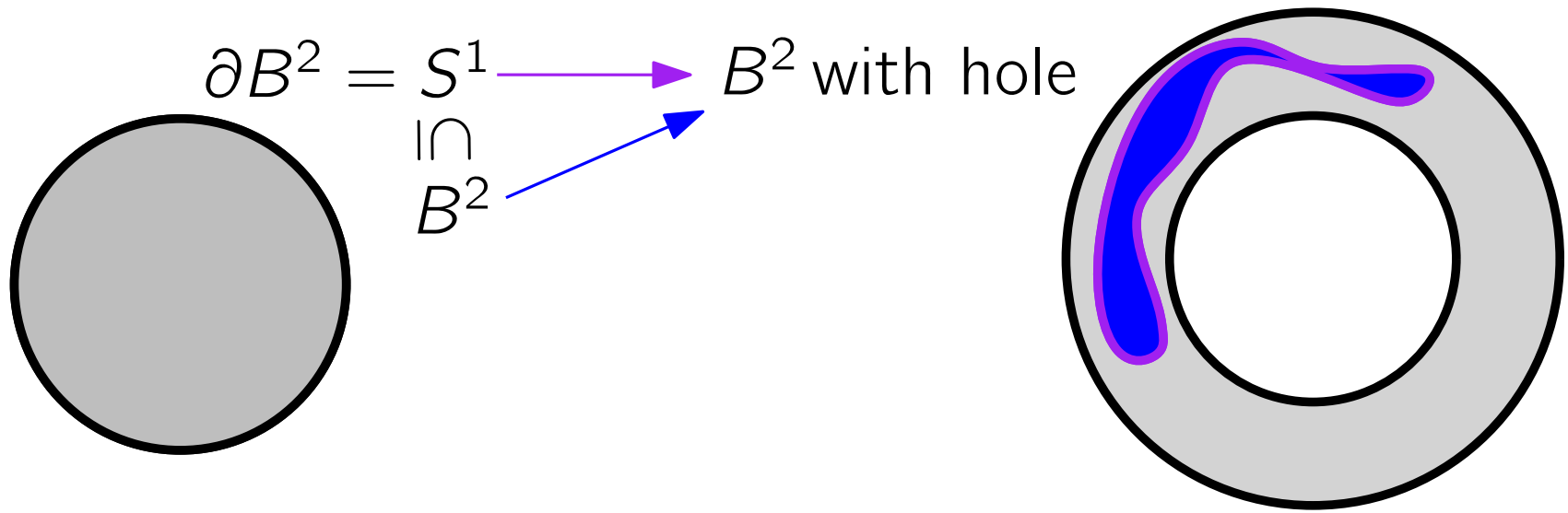

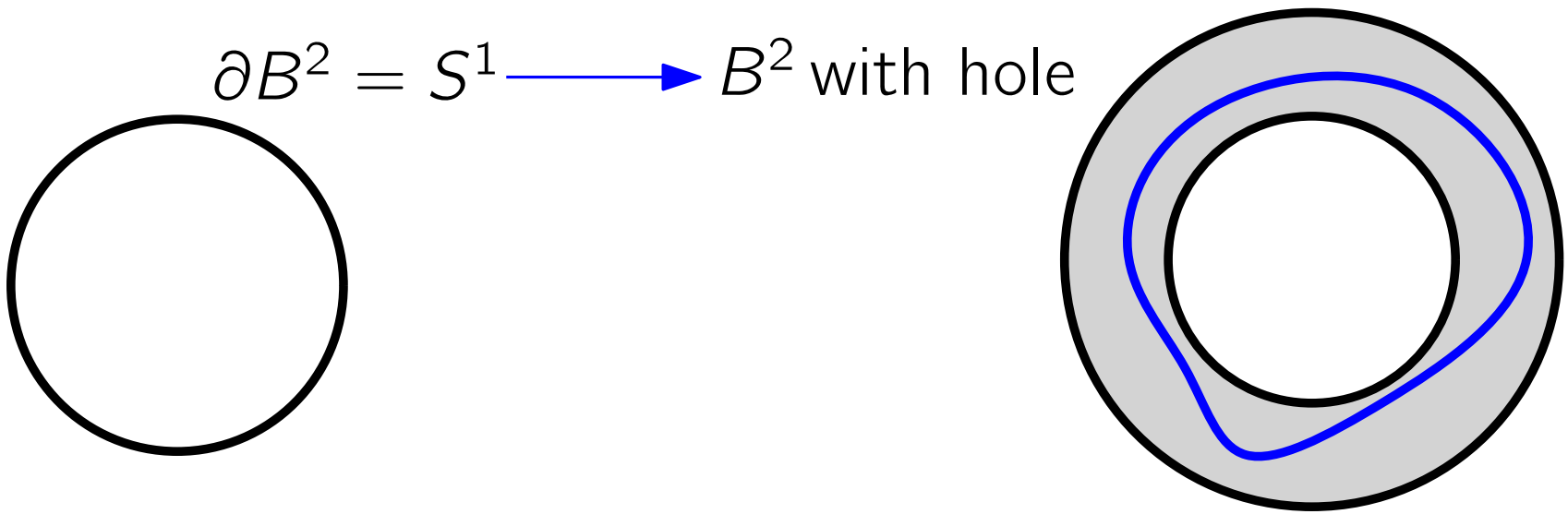

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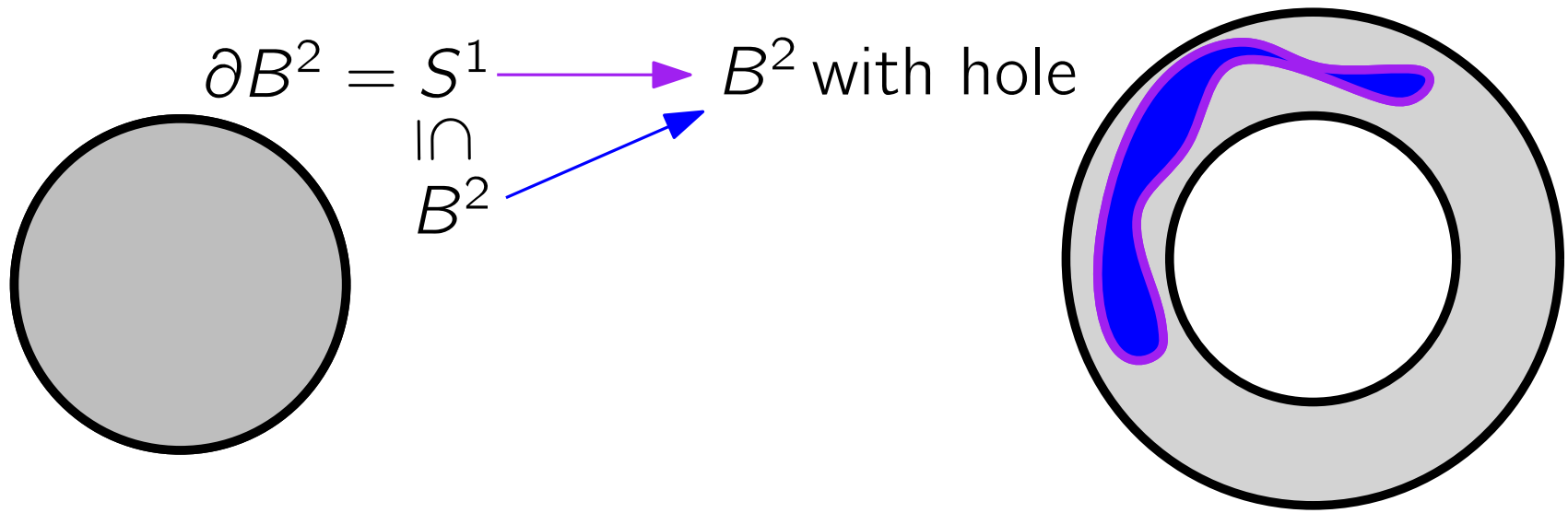

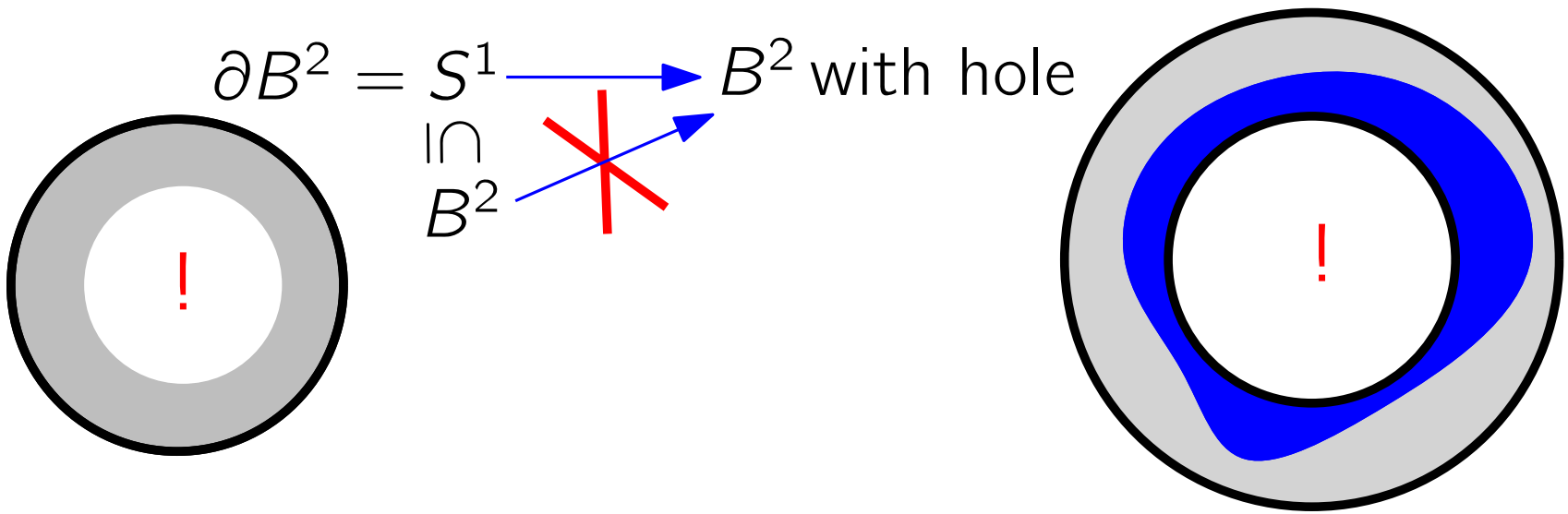

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Is $(c(1), c(2), c(3), c(4), c(5))$ extensible to every triangle?

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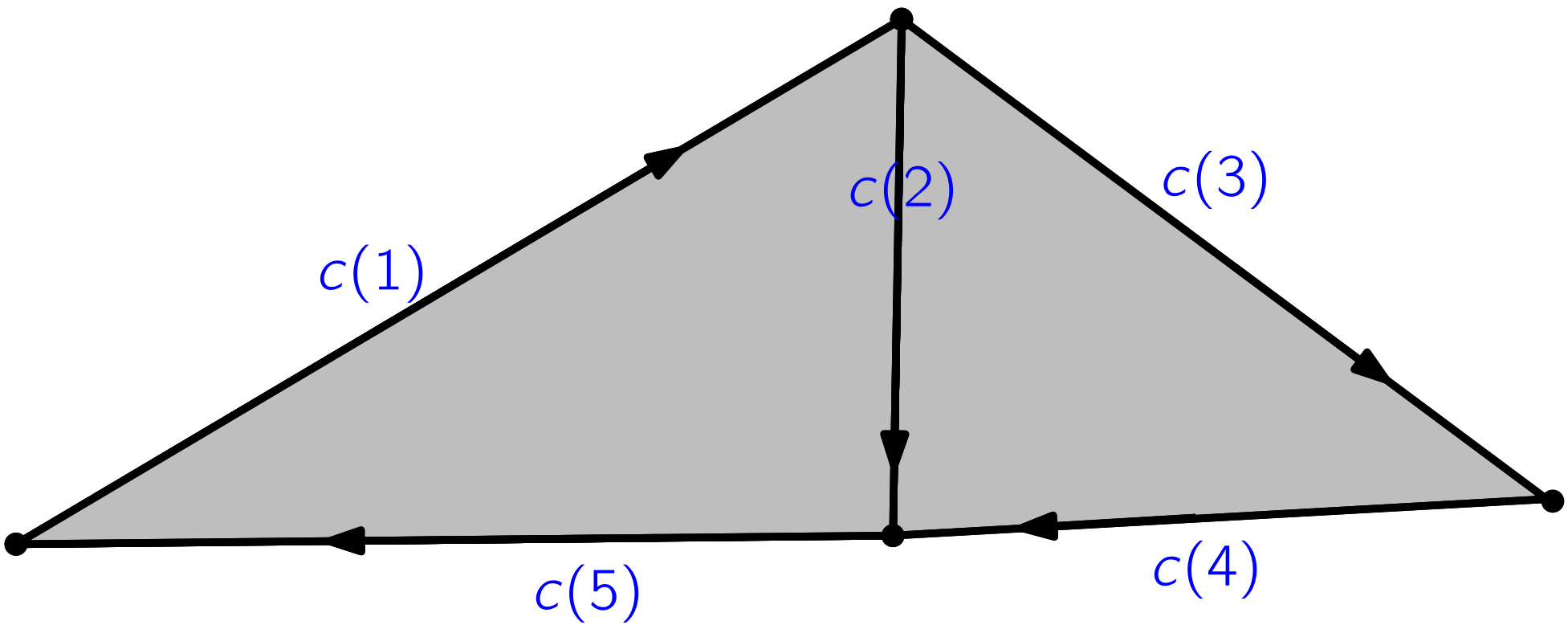

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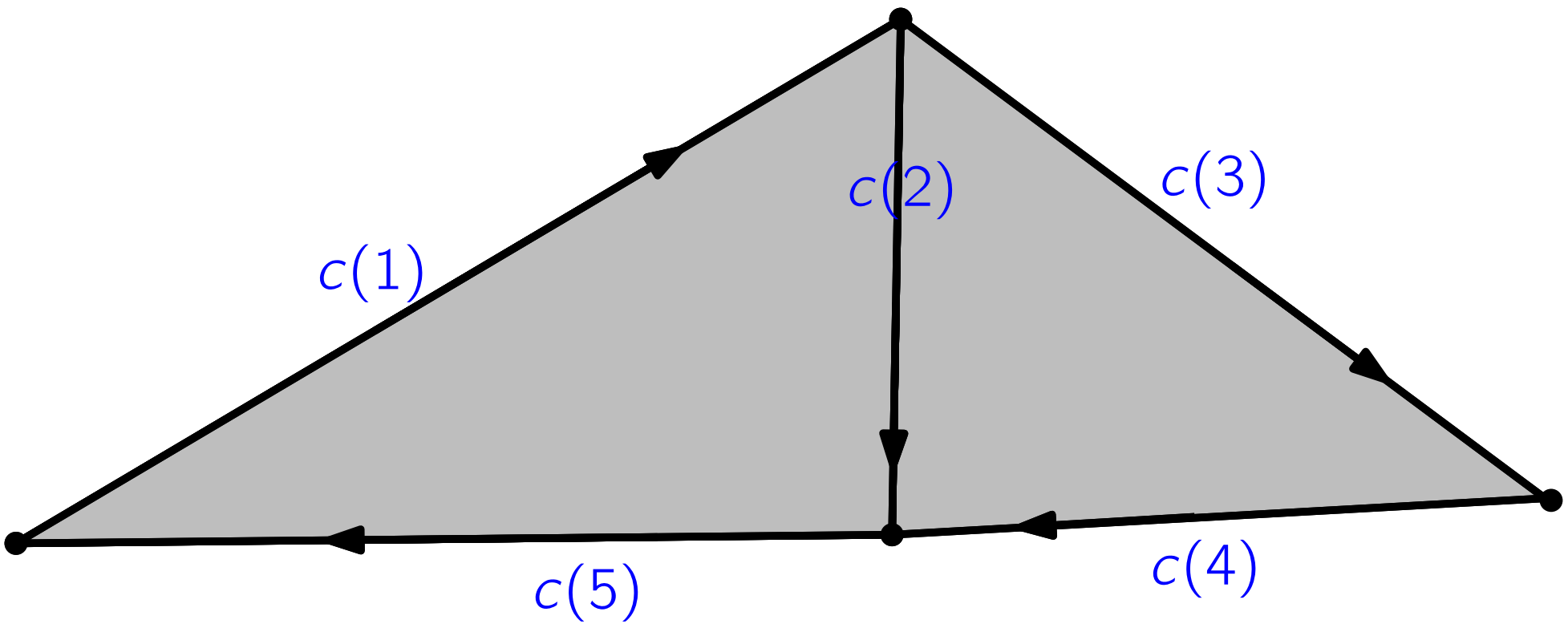

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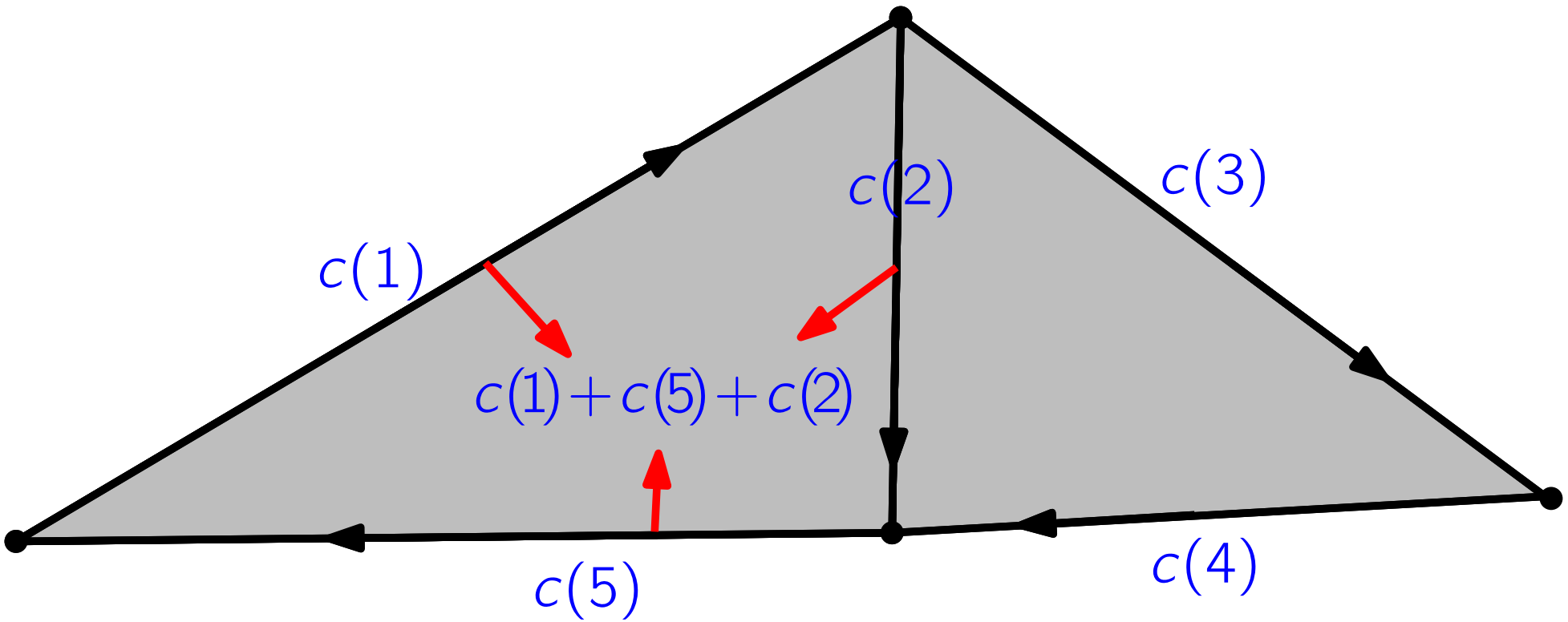

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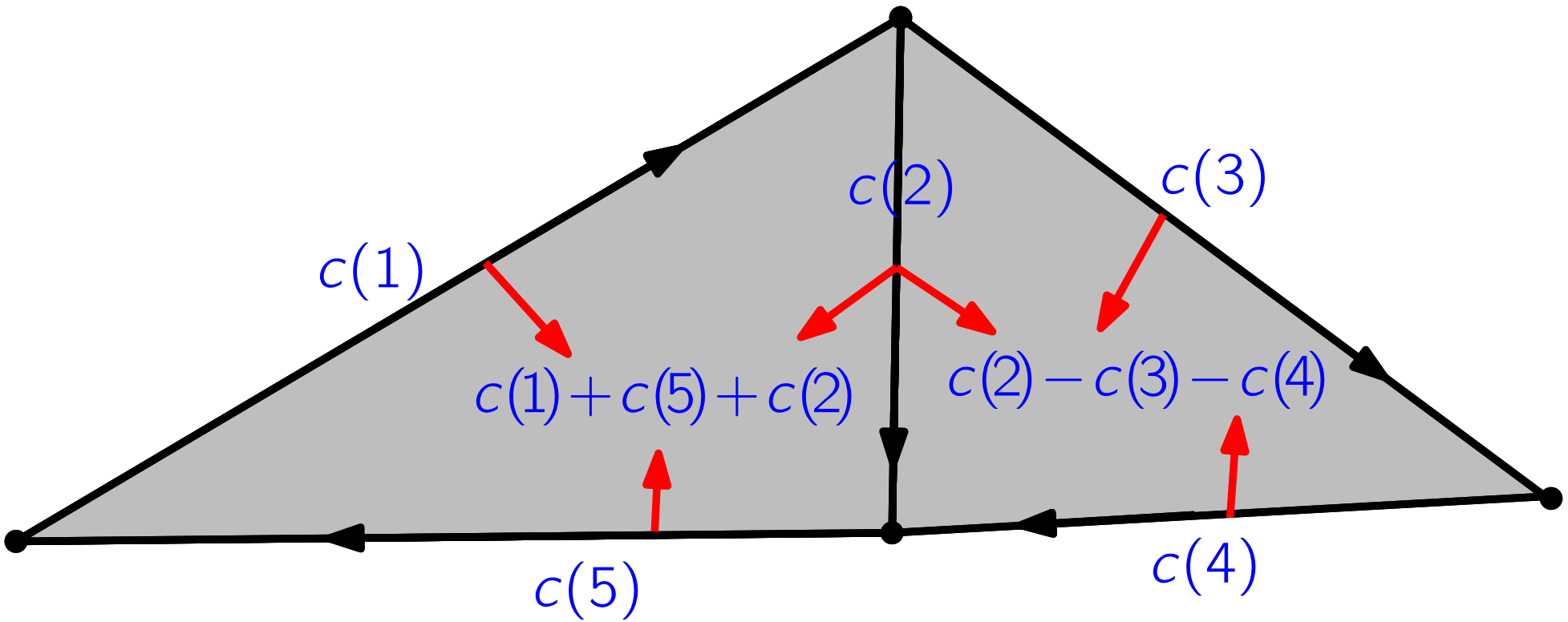

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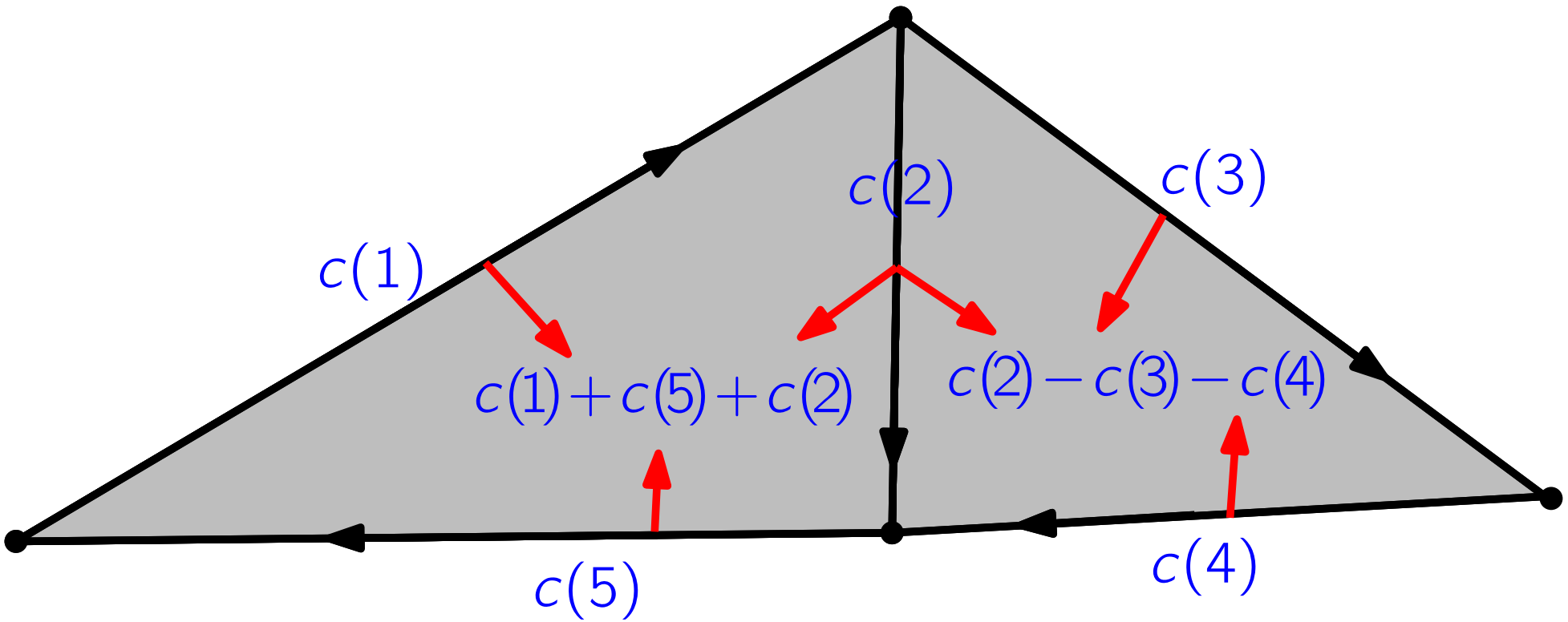

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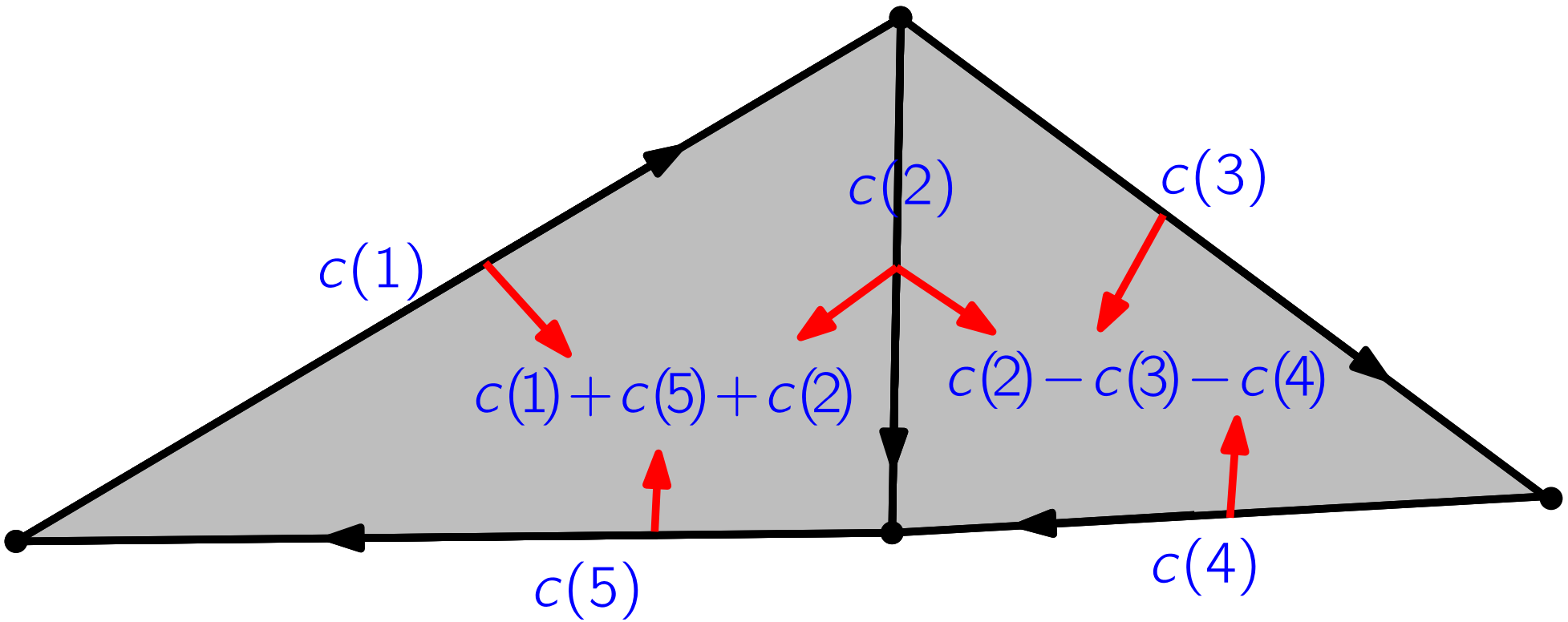

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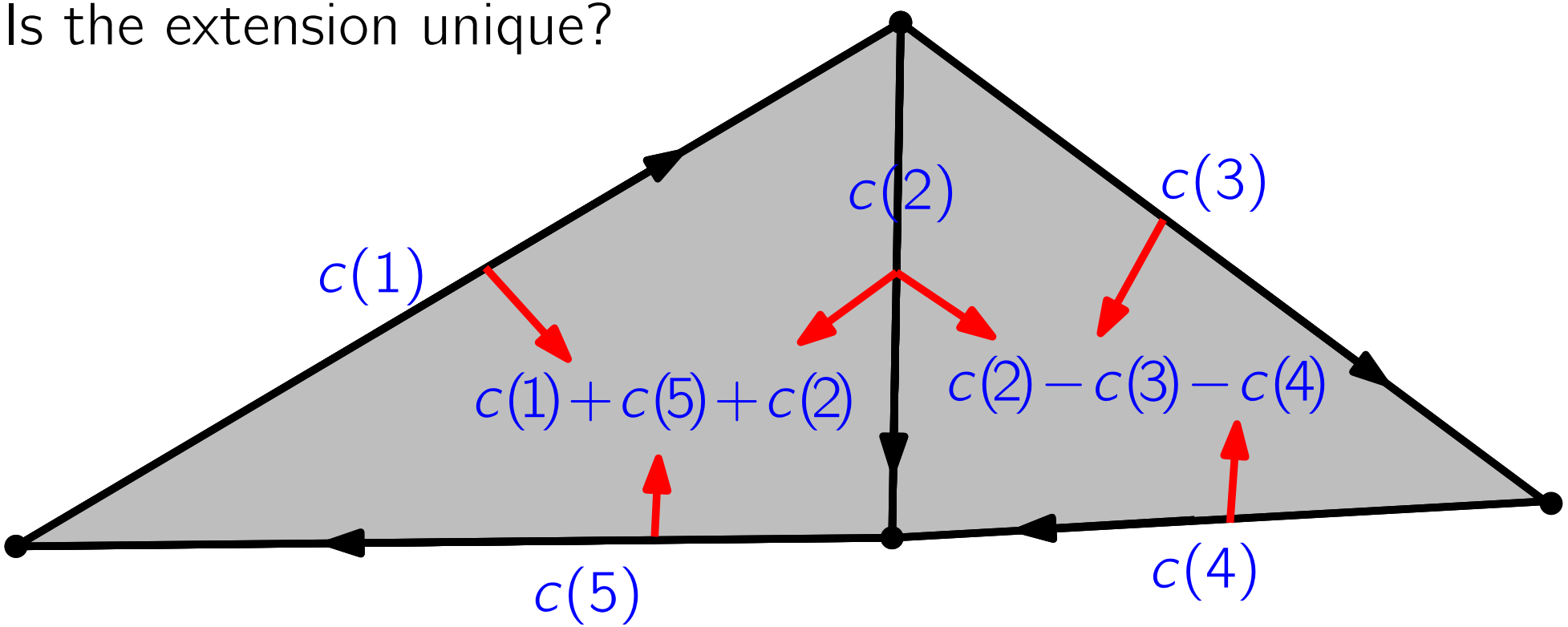

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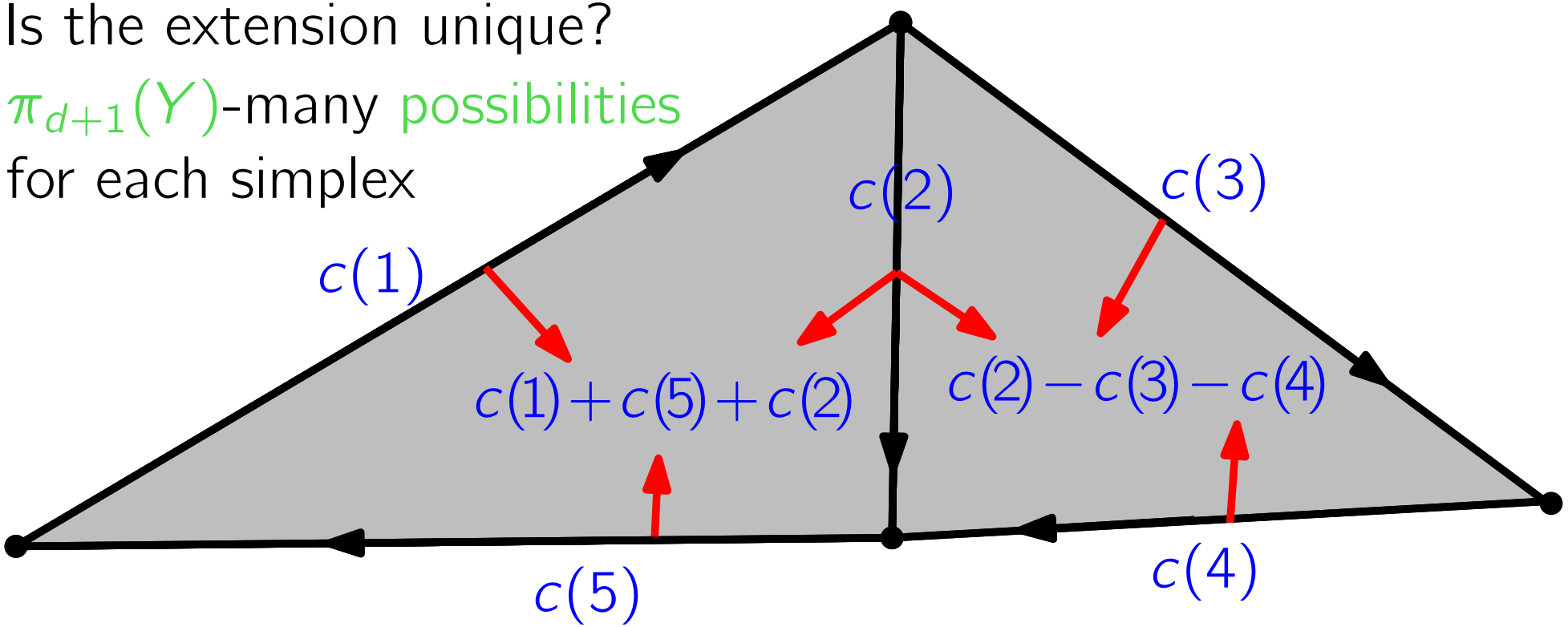

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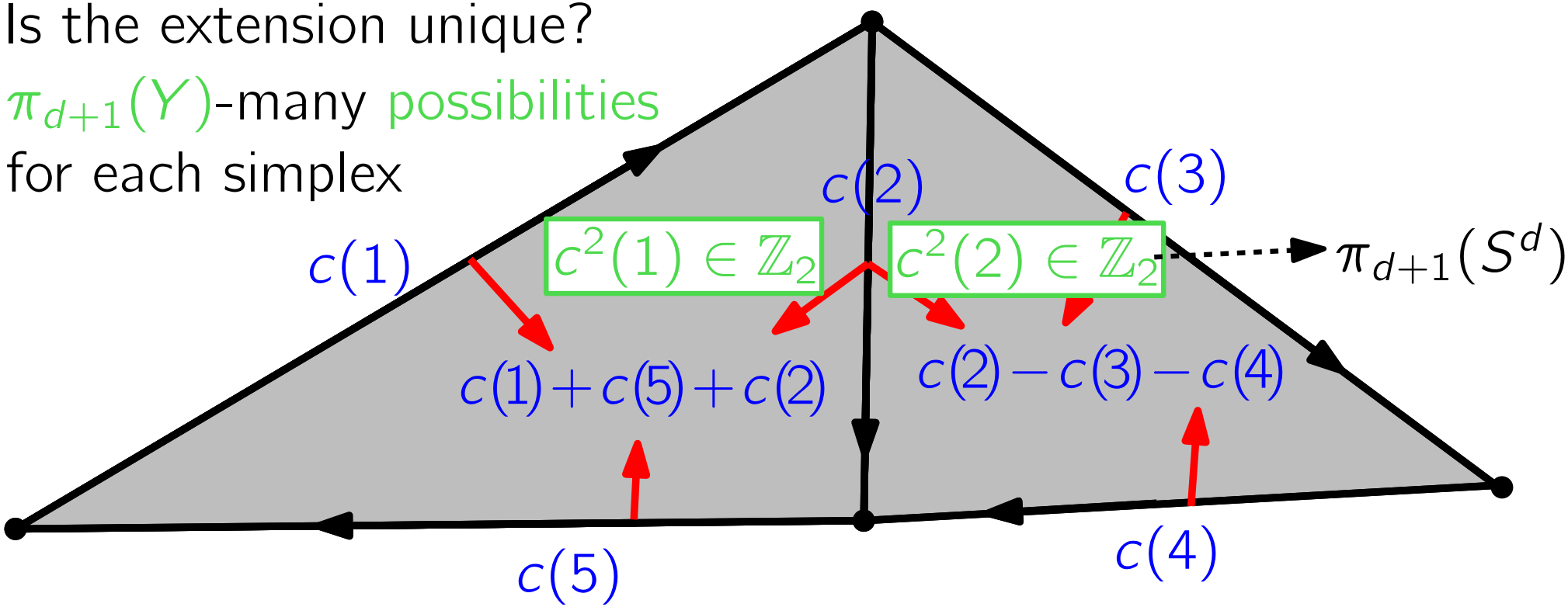

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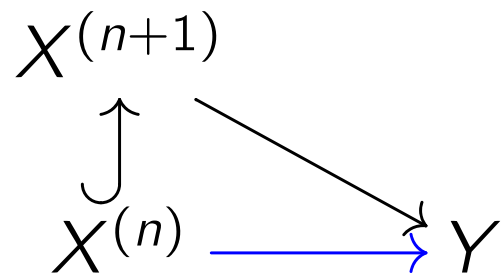

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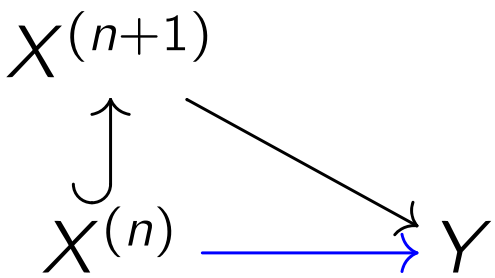

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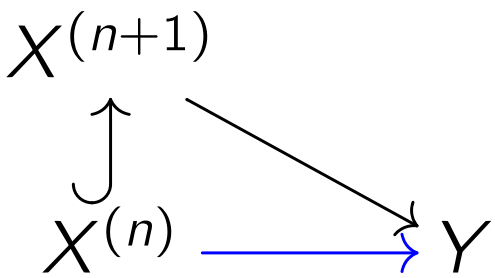

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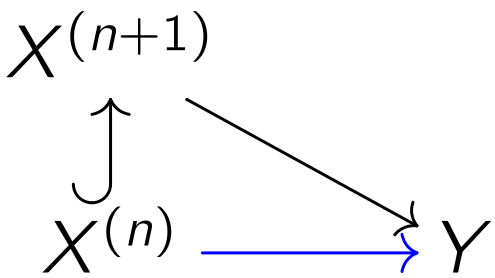

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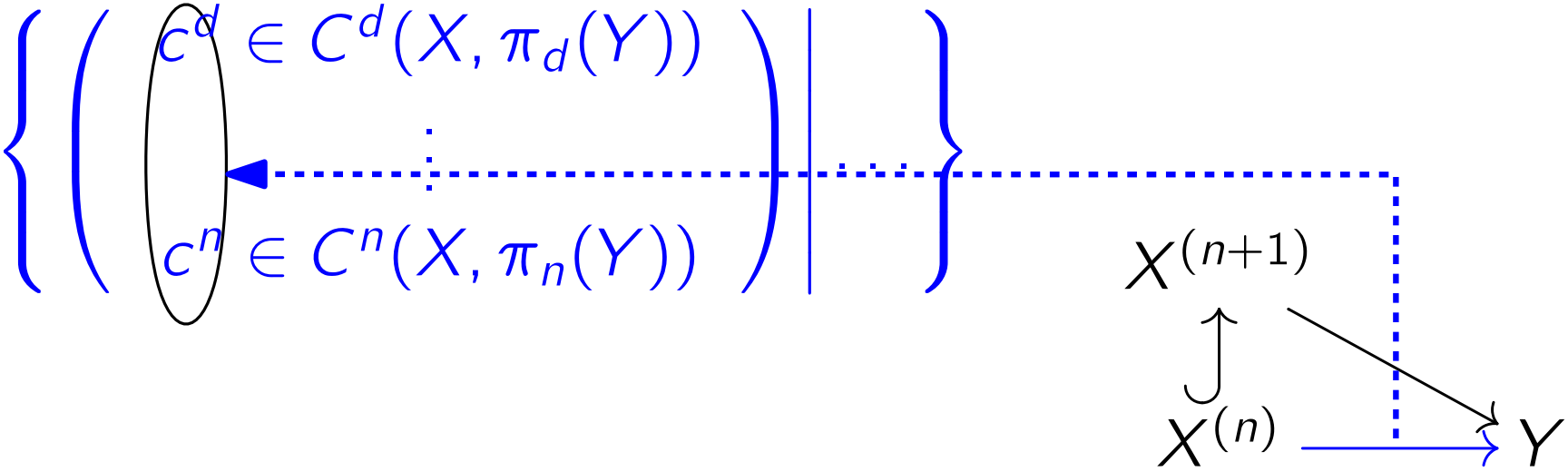

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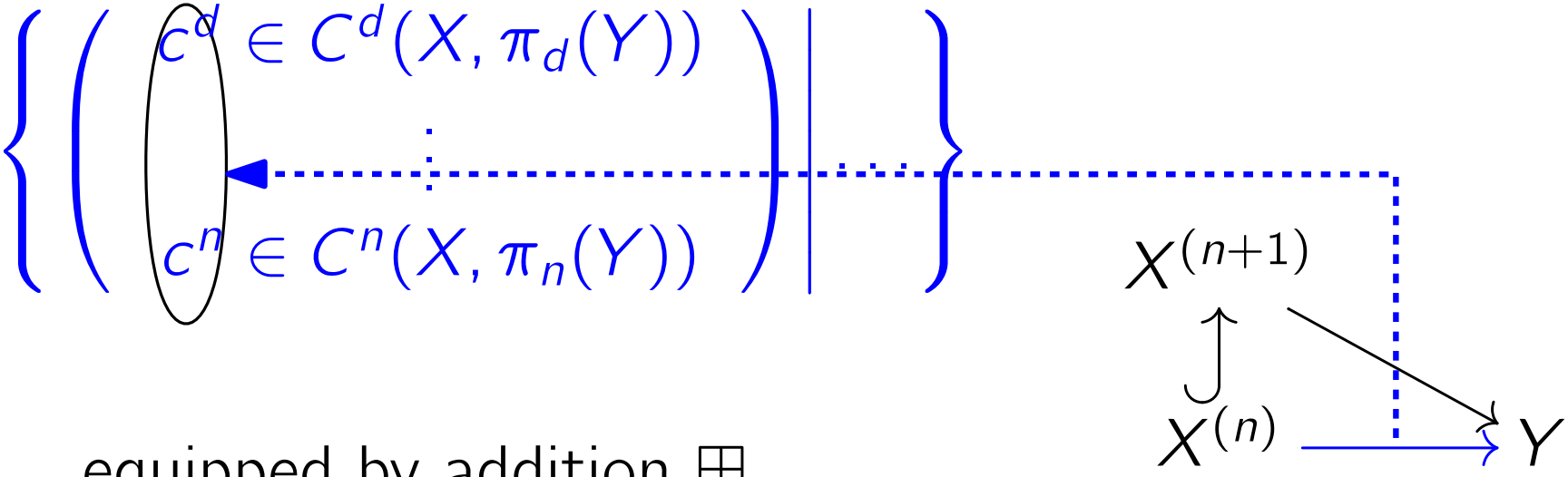

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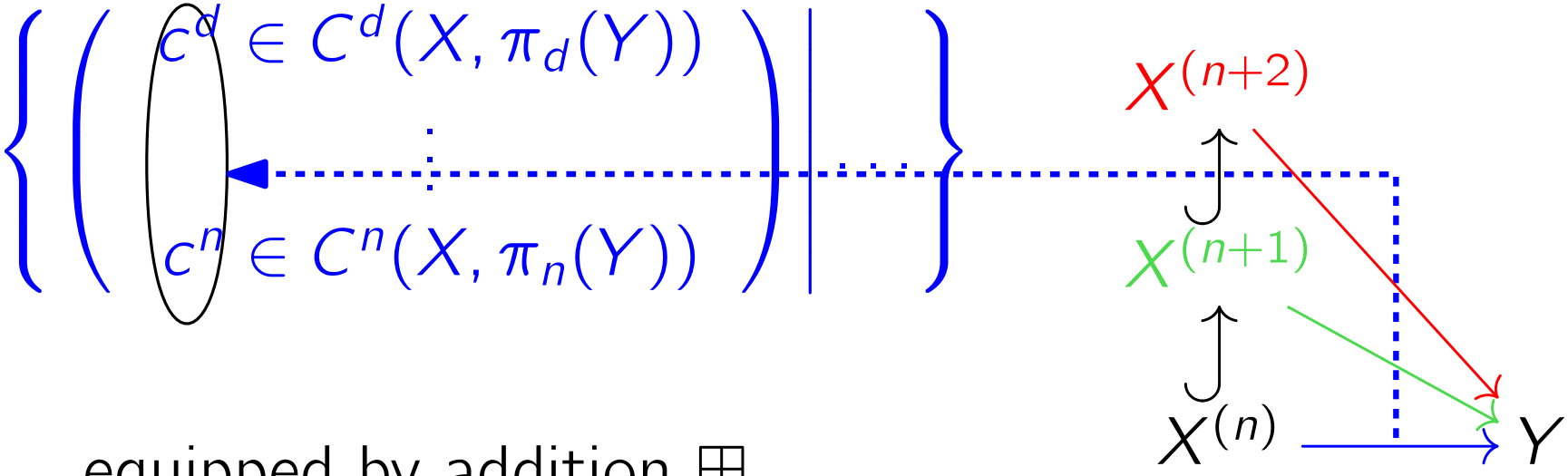

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- Which extensions

can be further extended?

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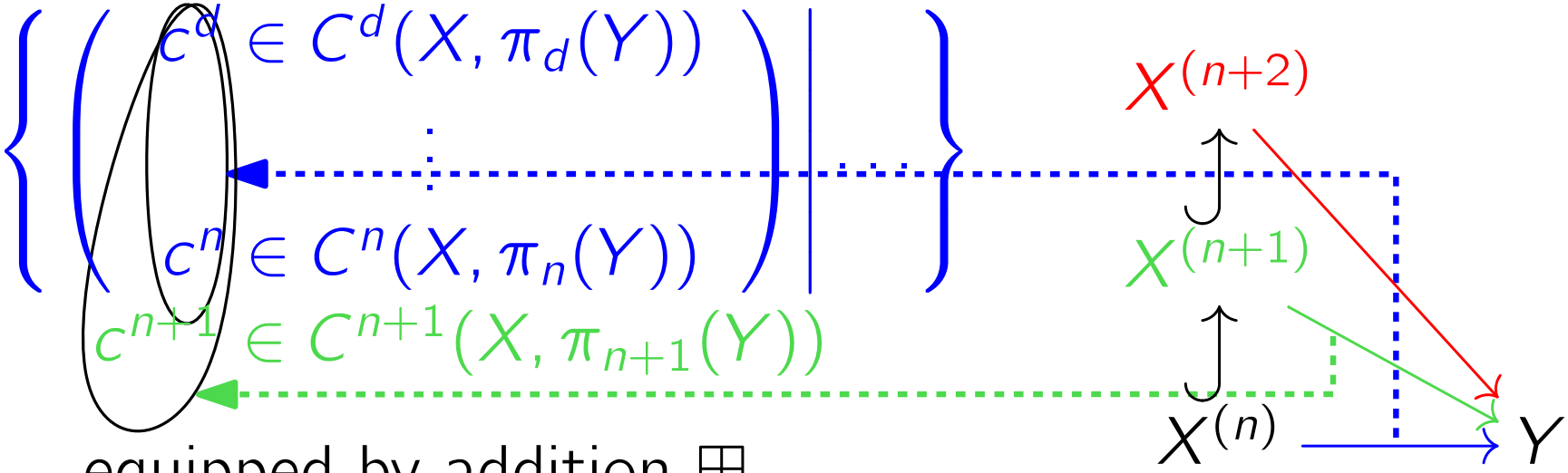

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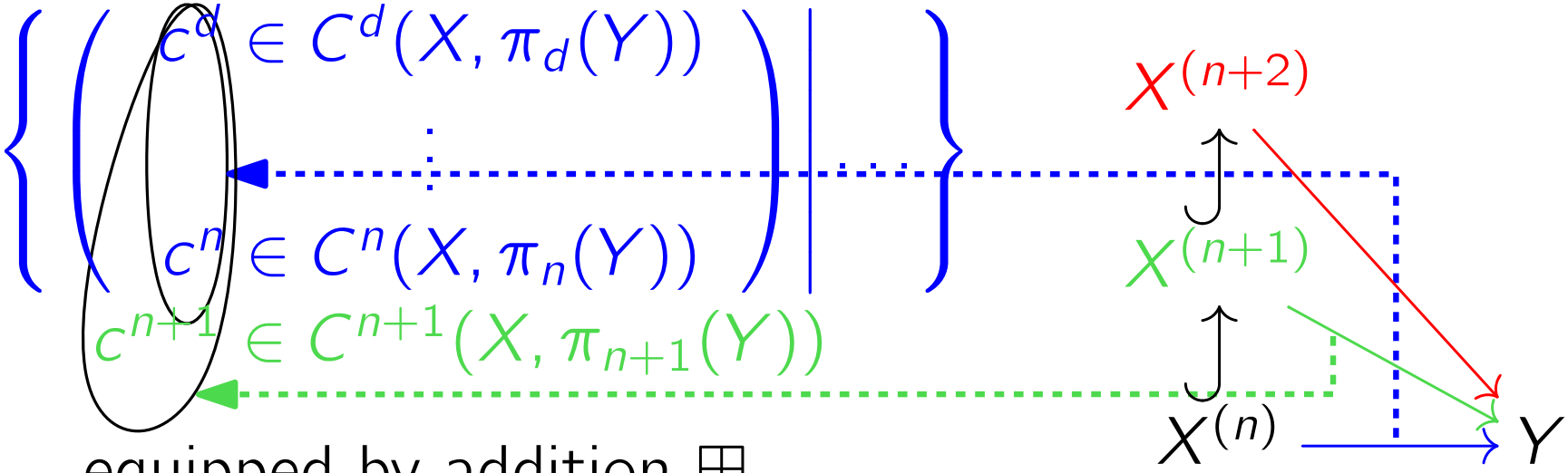

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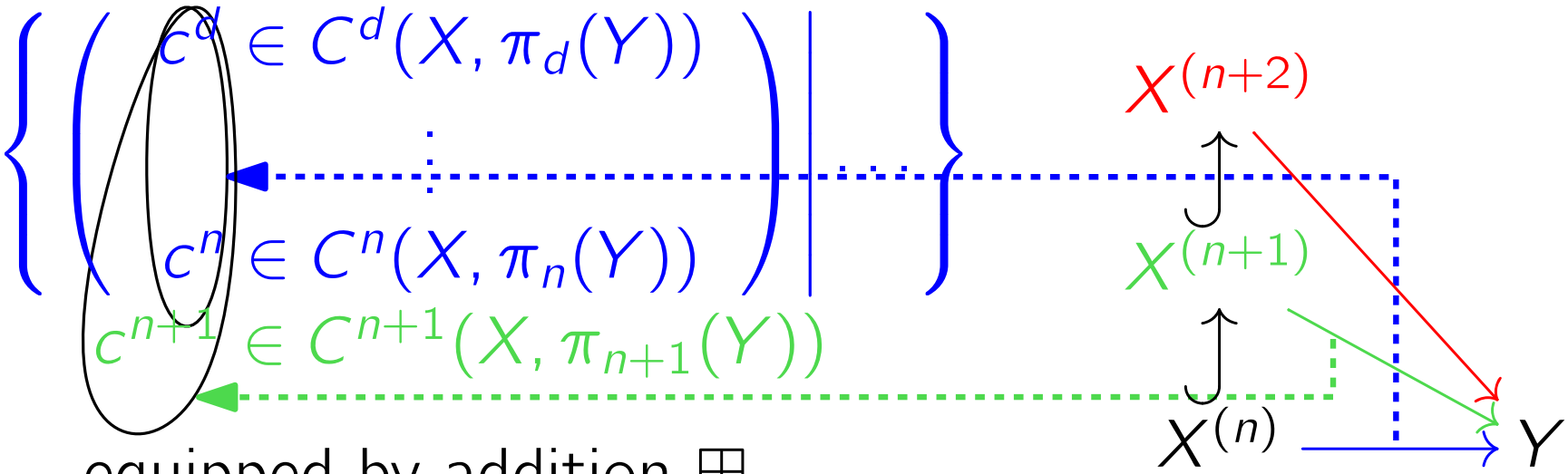

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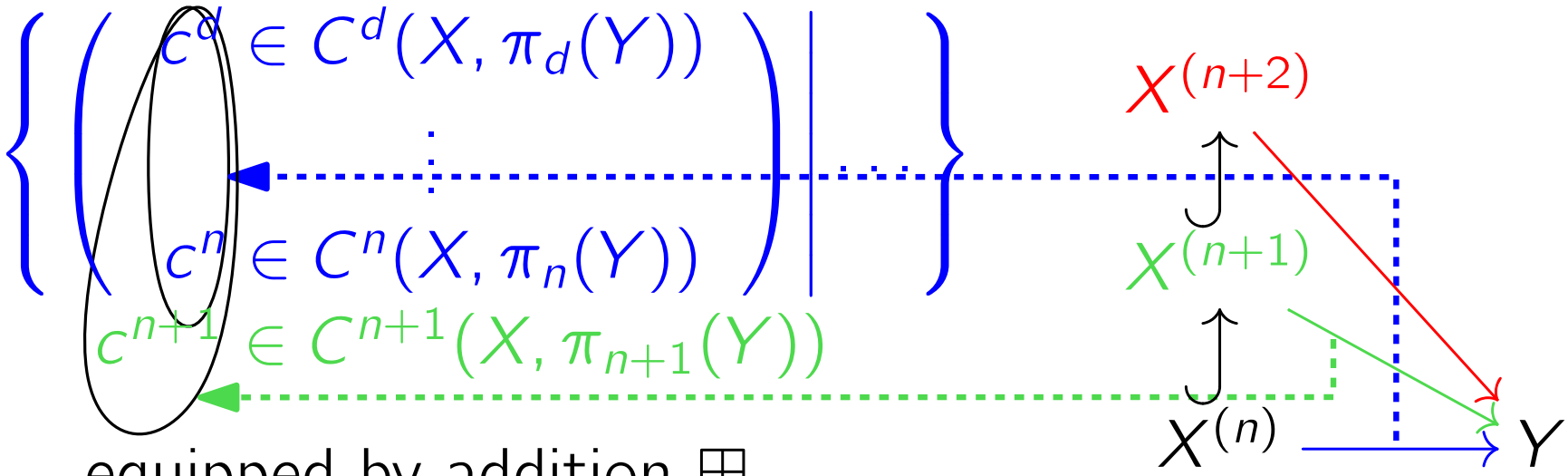

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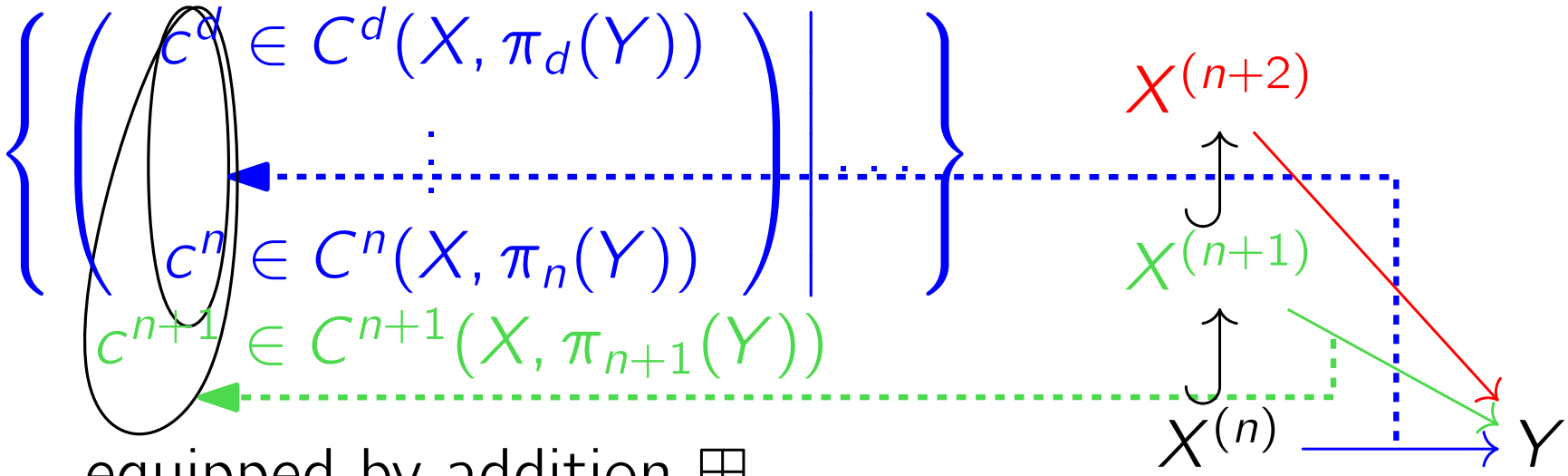

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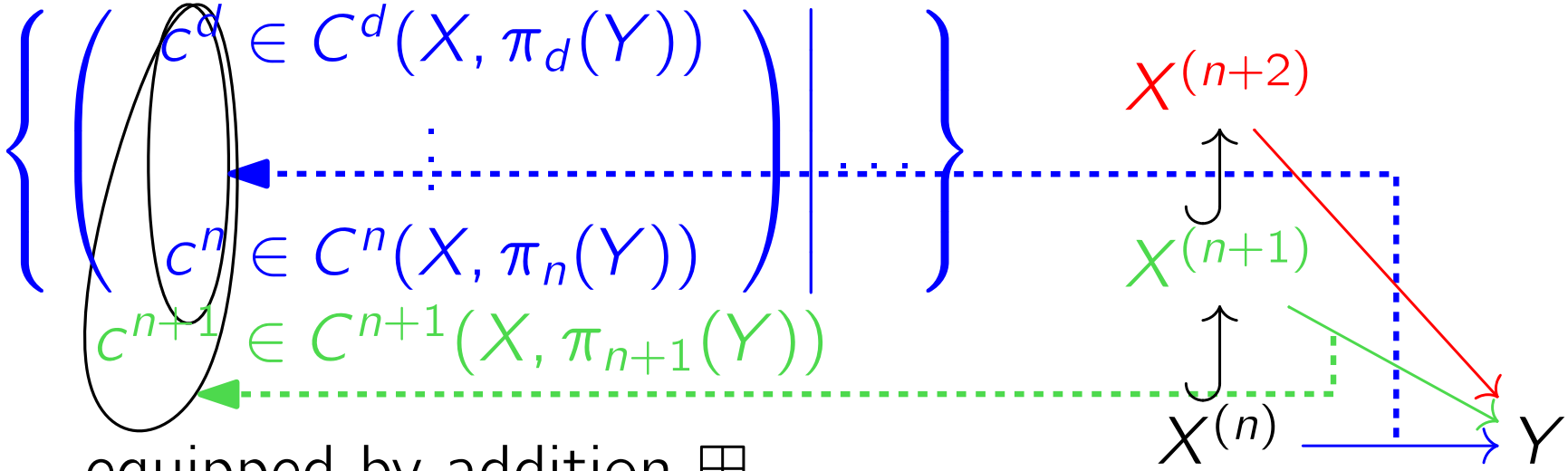

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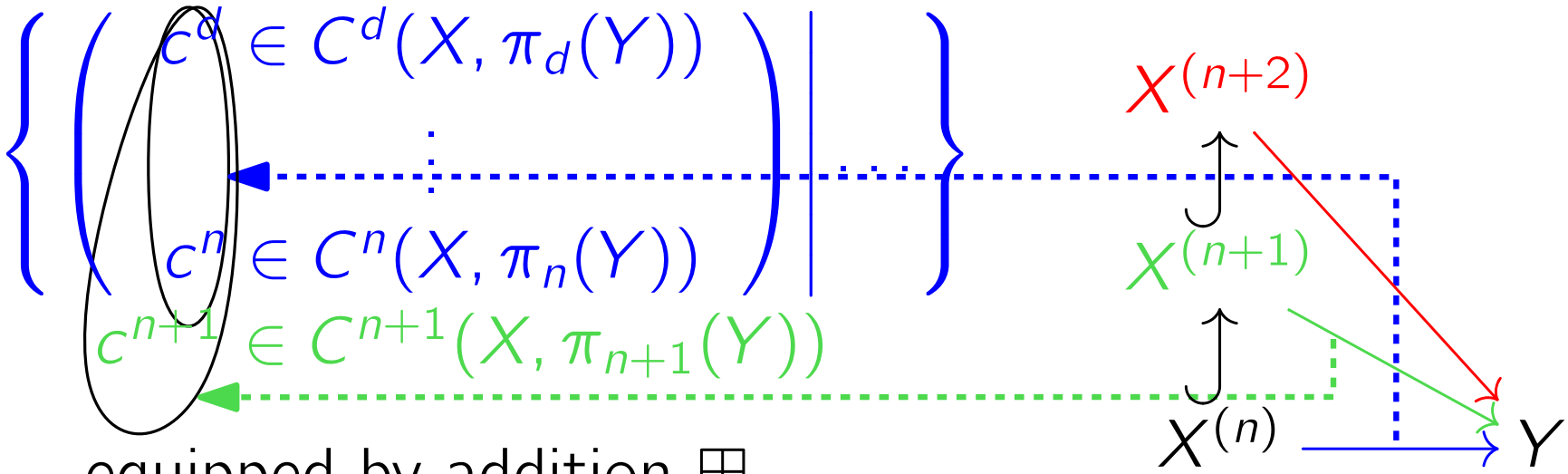

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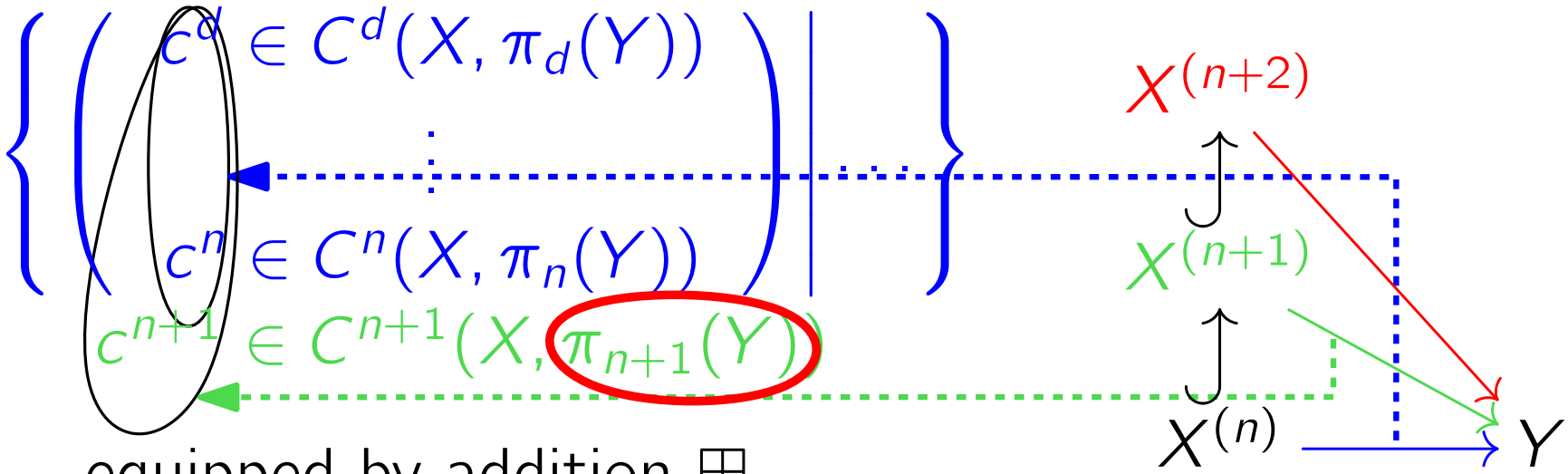

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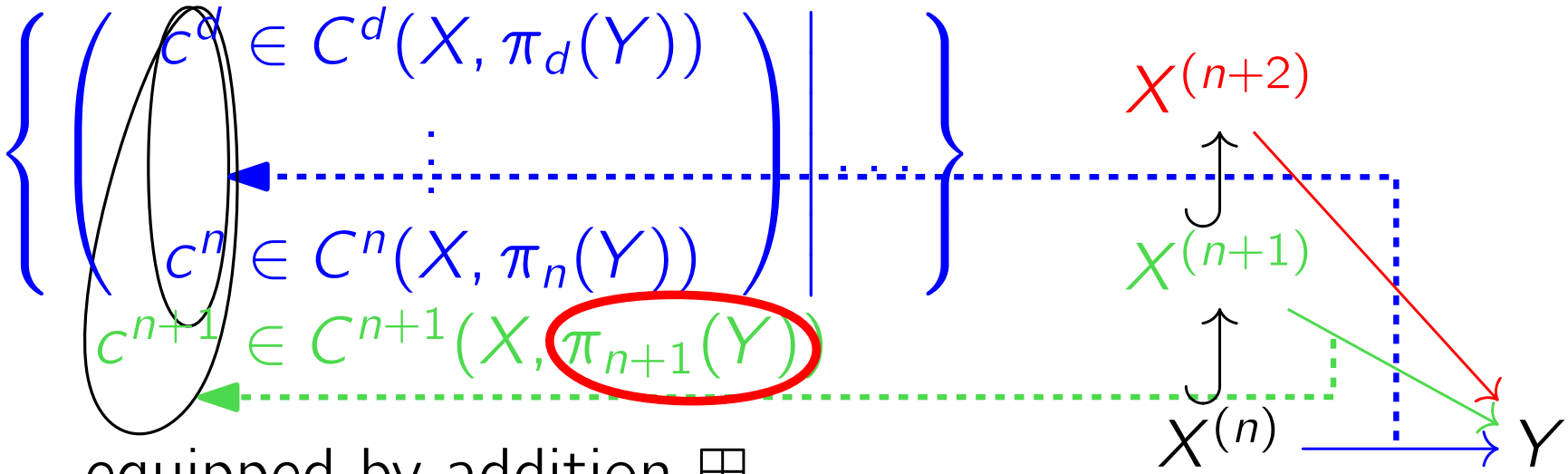

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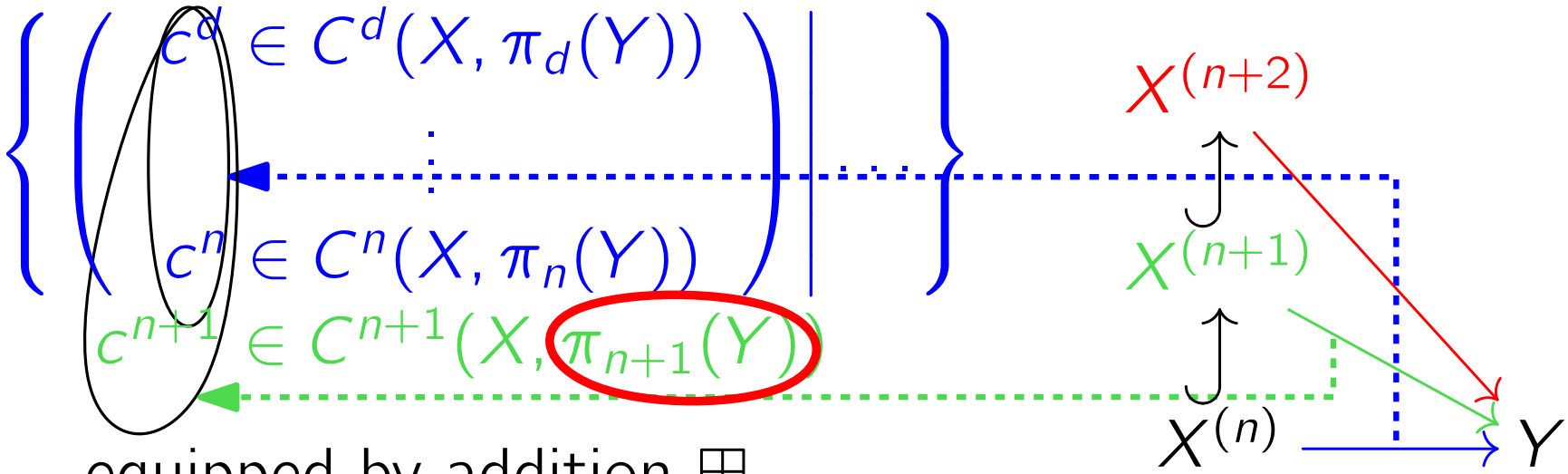

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Convenient to replace Y by its nth **Postnikov stage** P_n:

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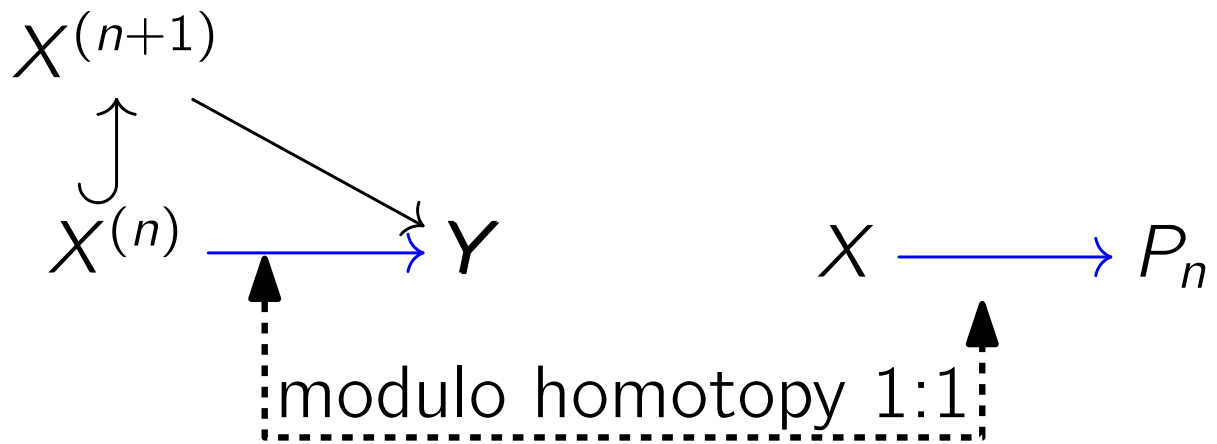

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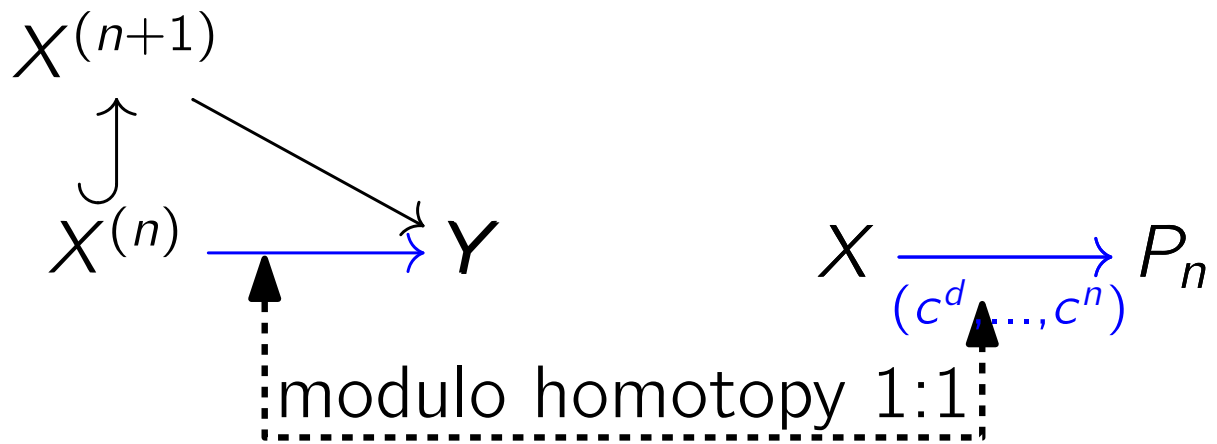

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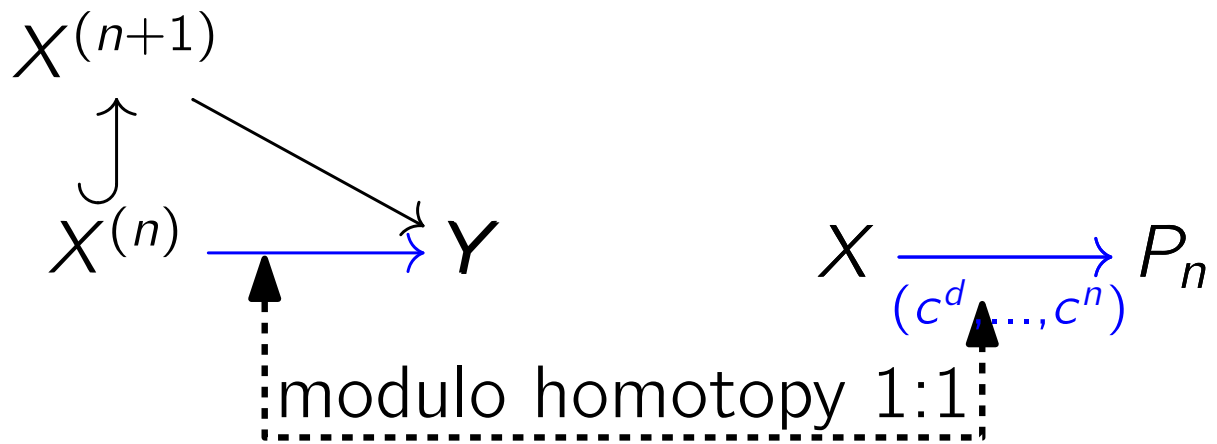

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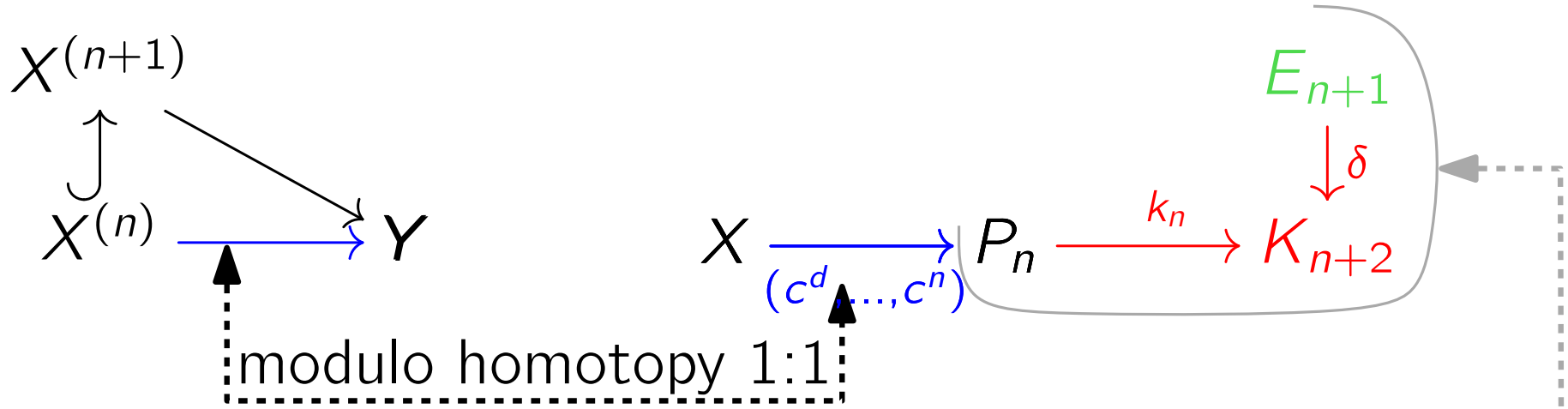

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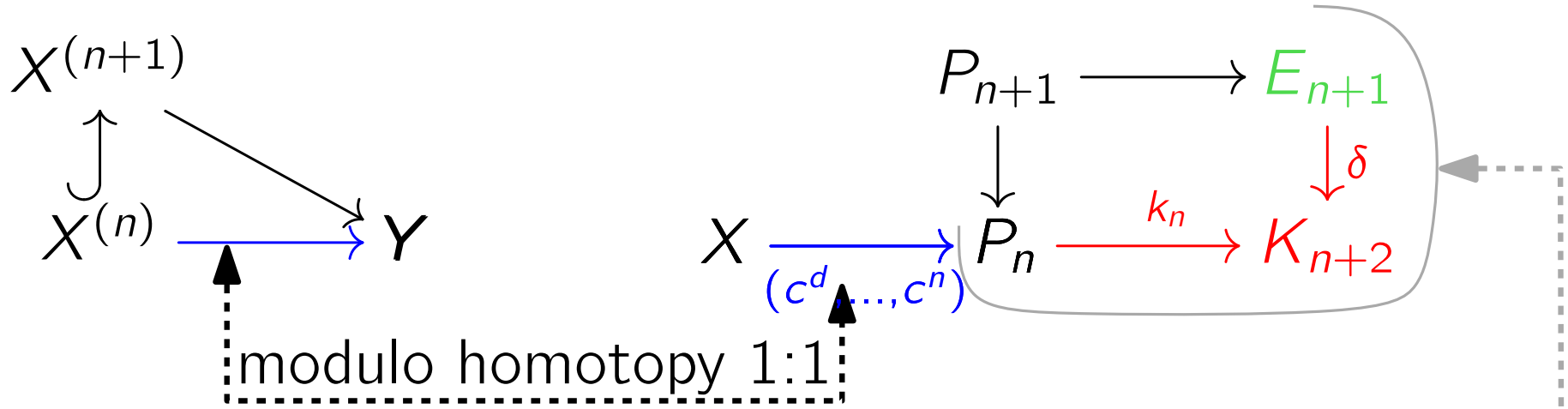

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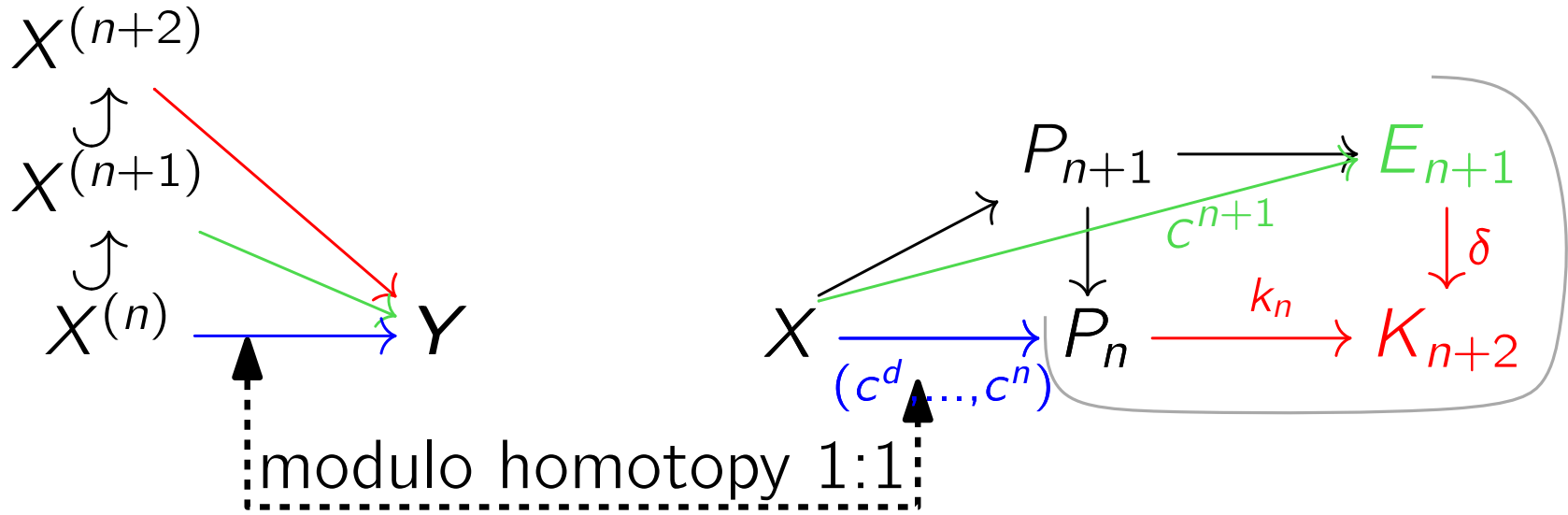

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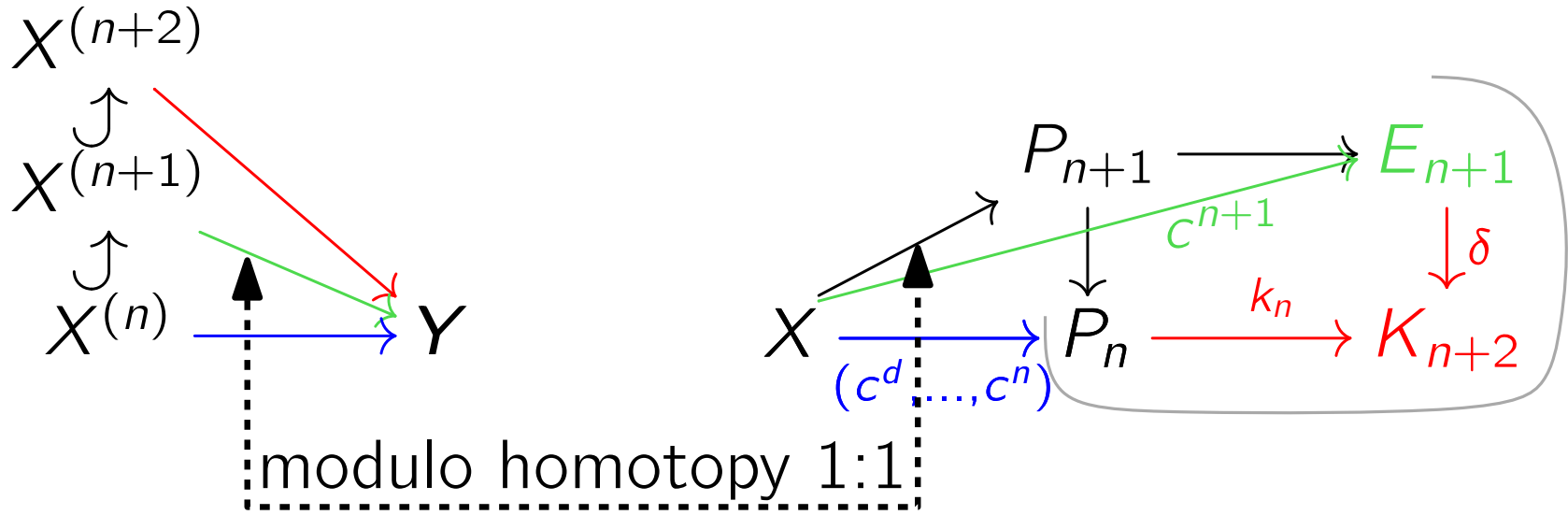

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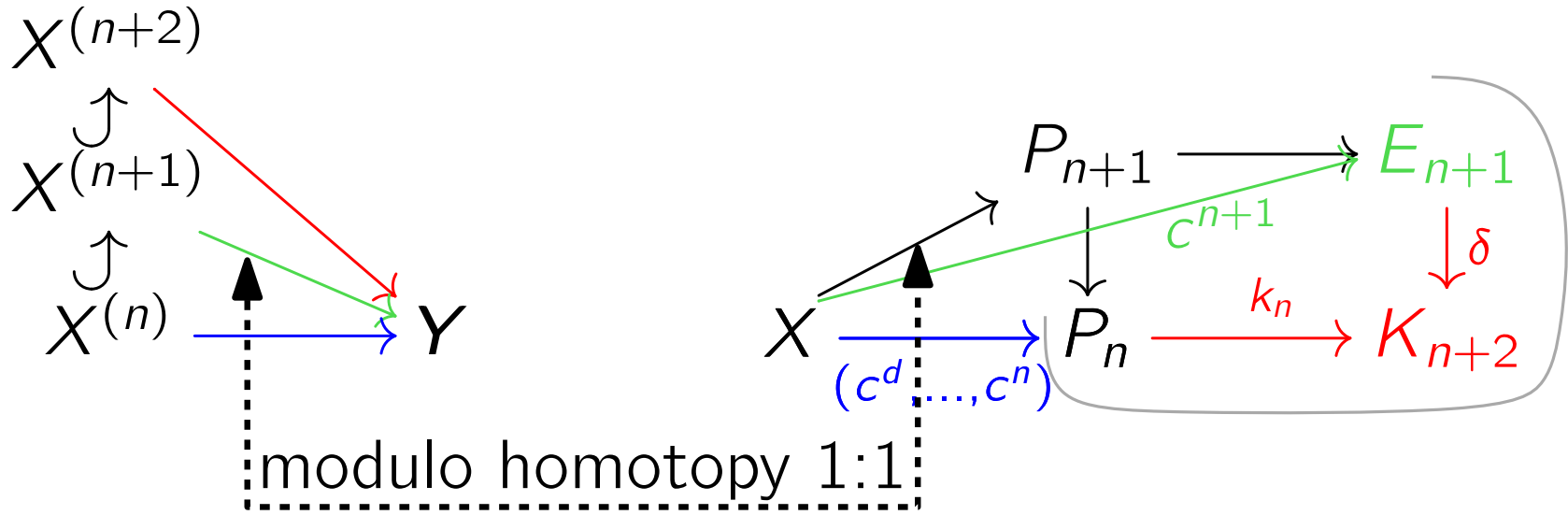

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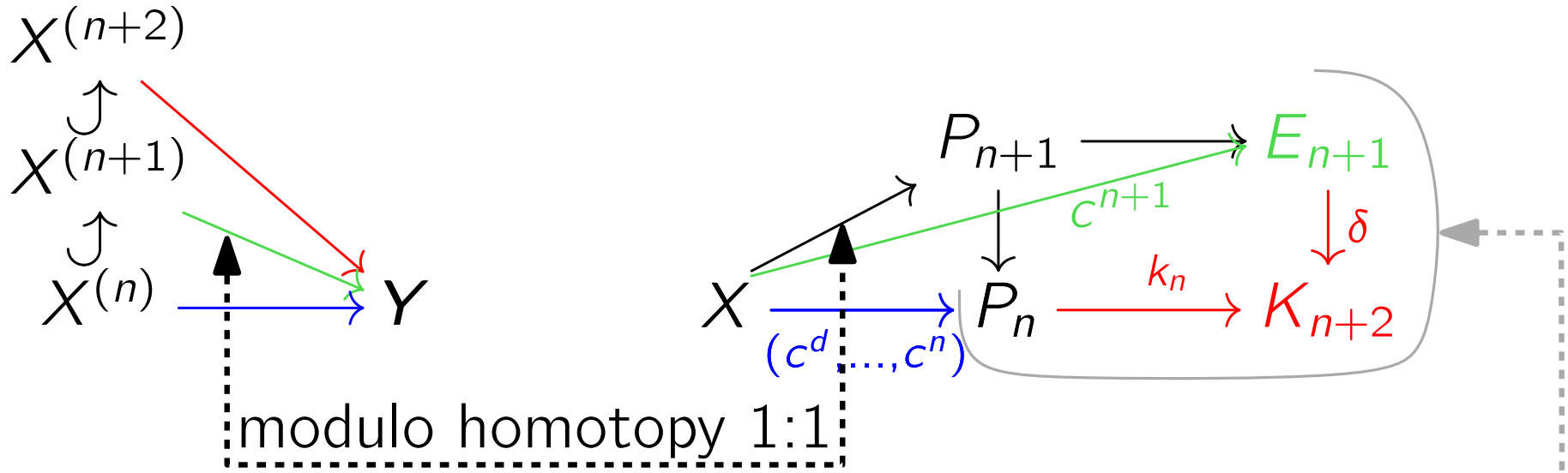

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k_n obtainable by computations in homology BUT P_n is infinite!

Effective homology (simplified) [Sergeraert et al.]

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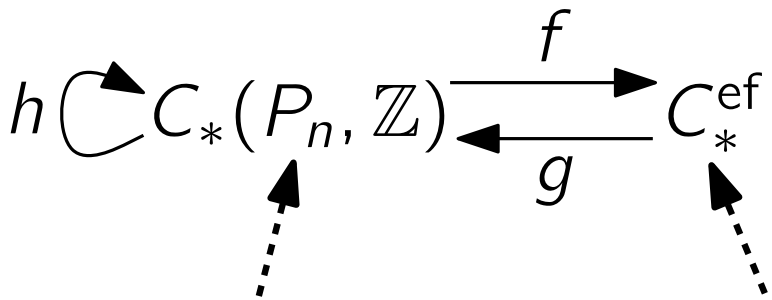

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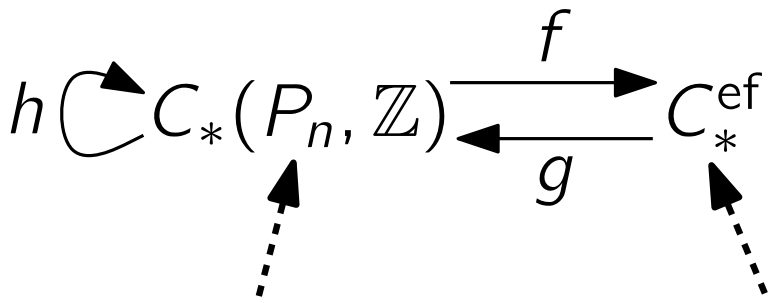

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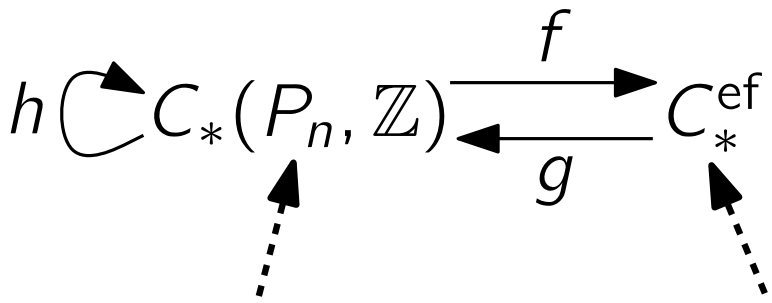

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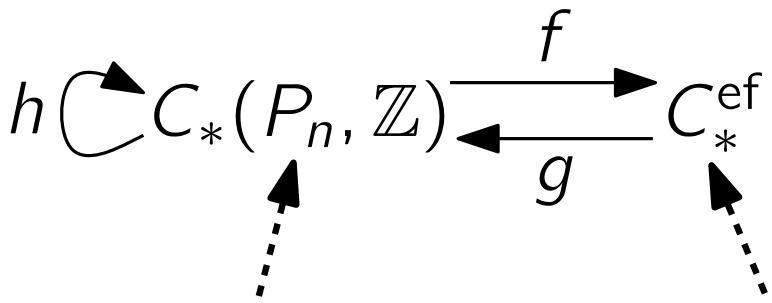

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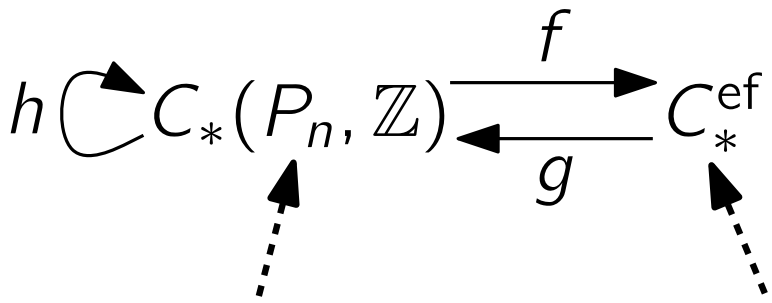

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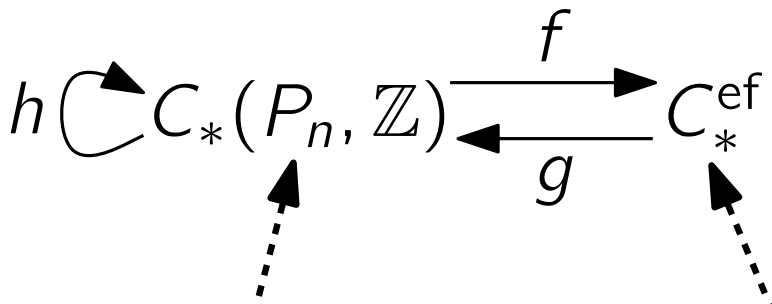

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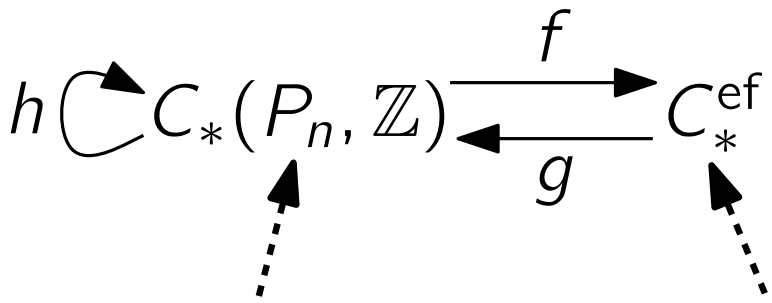

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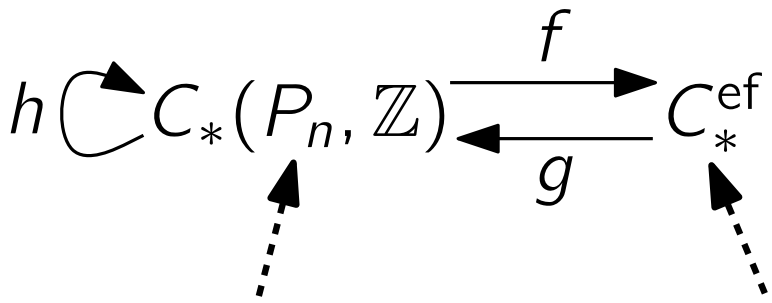

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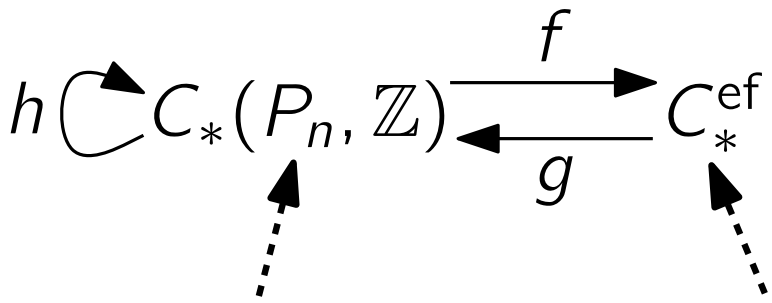

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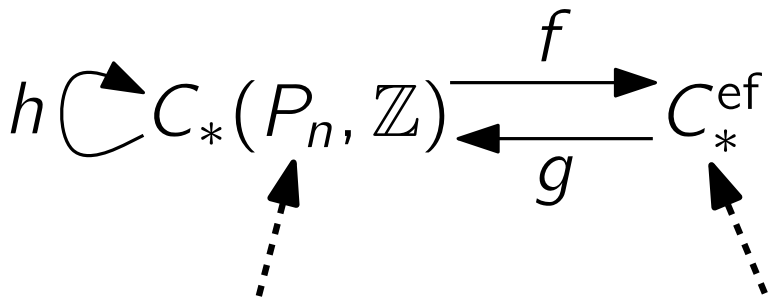

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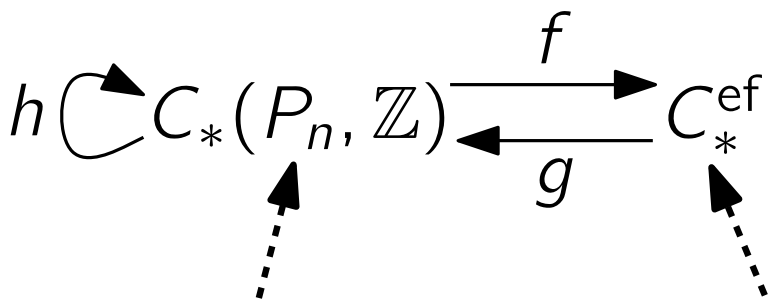
2) iterate "effective analogs" of simplicial constructions

   - product $\Downarrow$
   - classifying space
   - twisted product

Effective homology (simplified) [Sergeraert et al.]

- Effective homology for P_n is a homological reduction:



infinite simplicial set

∂, h and f given as algorithms

finitely generated

∂^{ef} given by matrices

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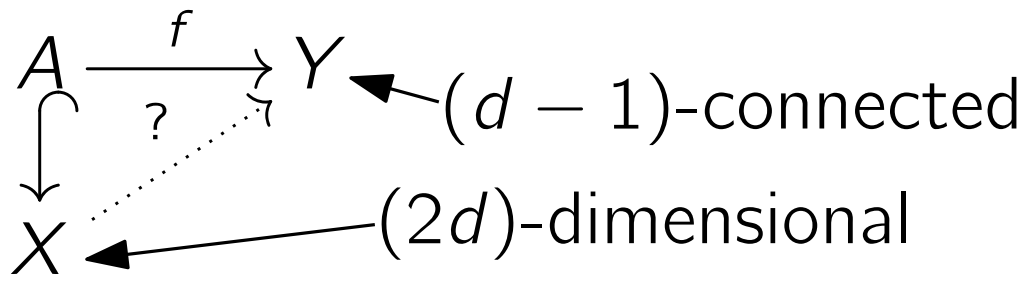
- **linear algebra over** $\mathbb{Z}$
  Smith normal form computation

Work in progress — the negative side

Extendability **undecidable** for



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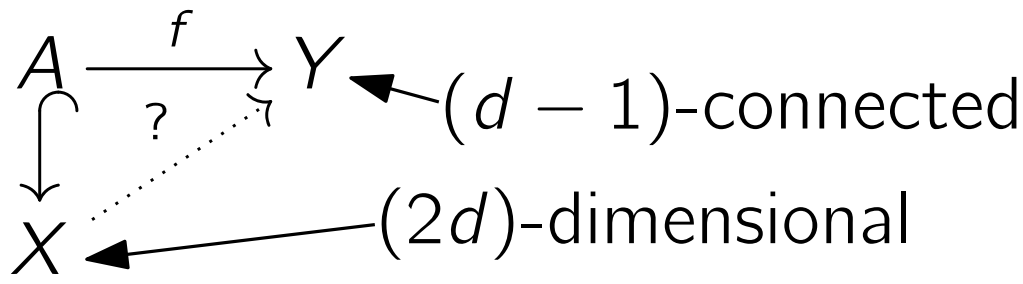

Proof: By reduction from

system of quadratic Diophantine eqns: $p_1(x_1, \ldots, x_r) = b_1$

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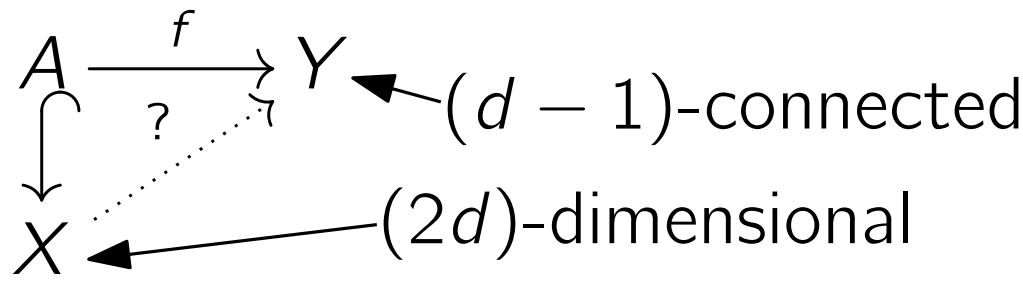

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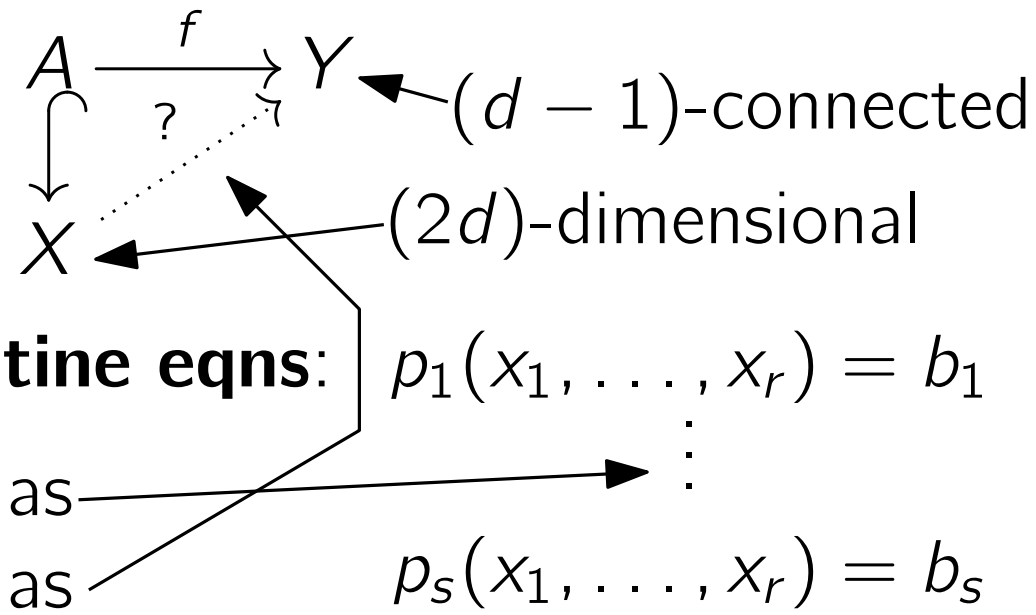

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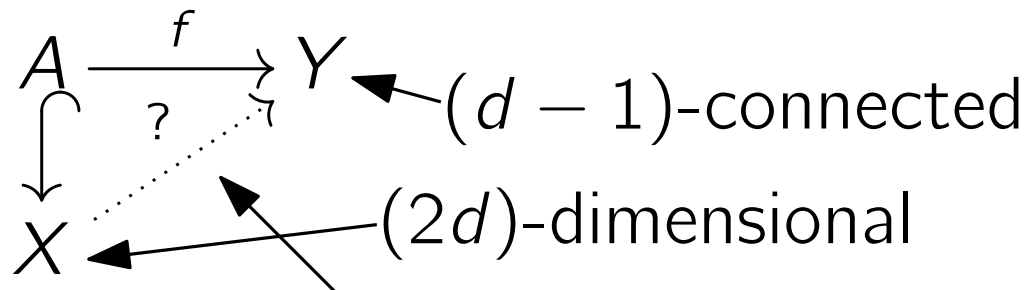

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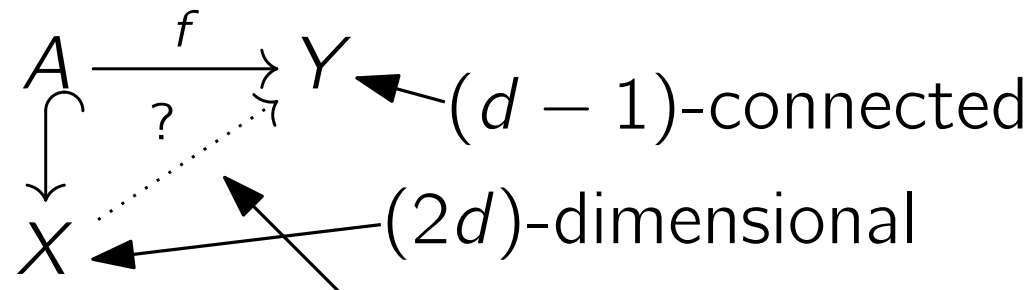

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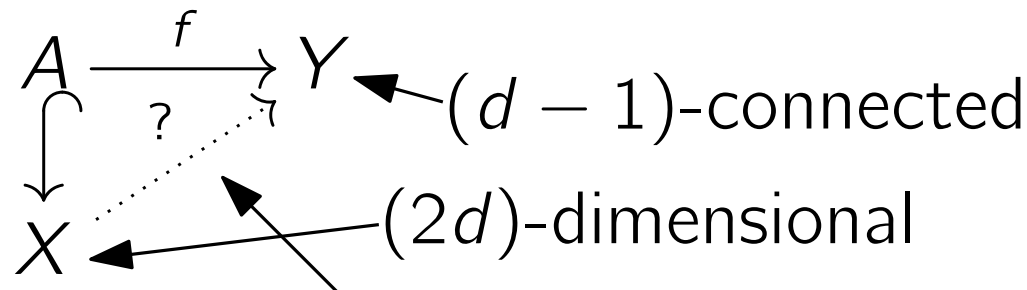

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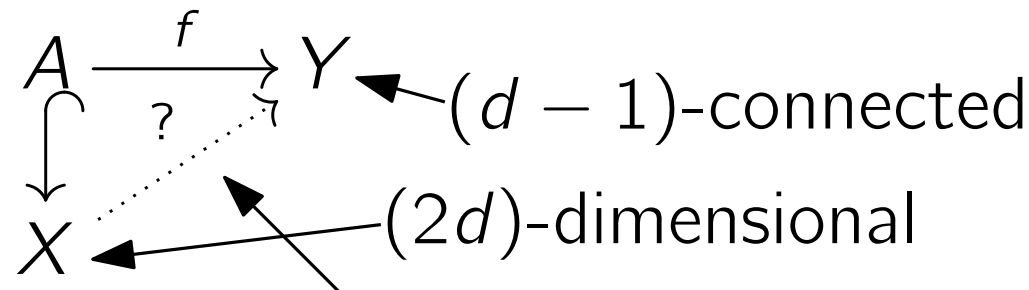

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  for n **fixed**...
  - $\pi_n(Y)$ can be computed in polynomial time
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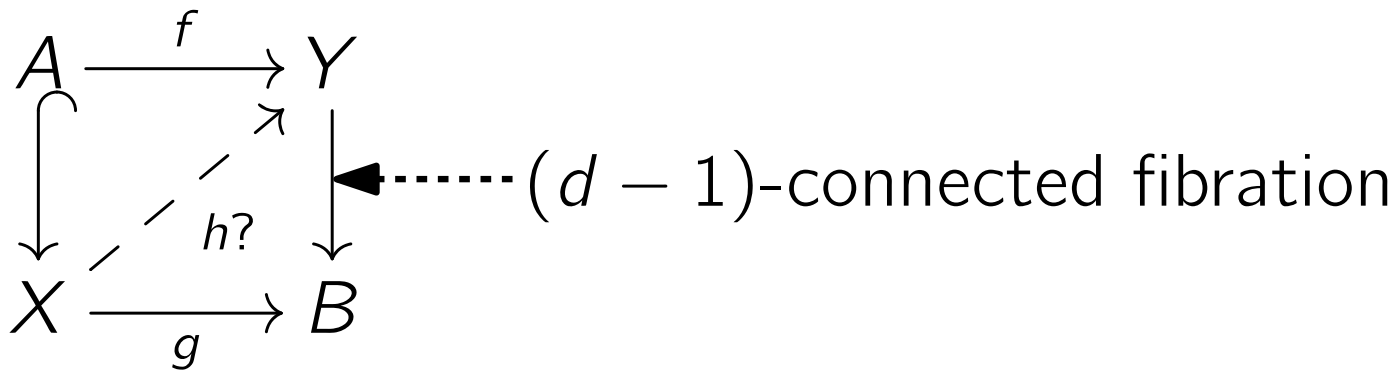

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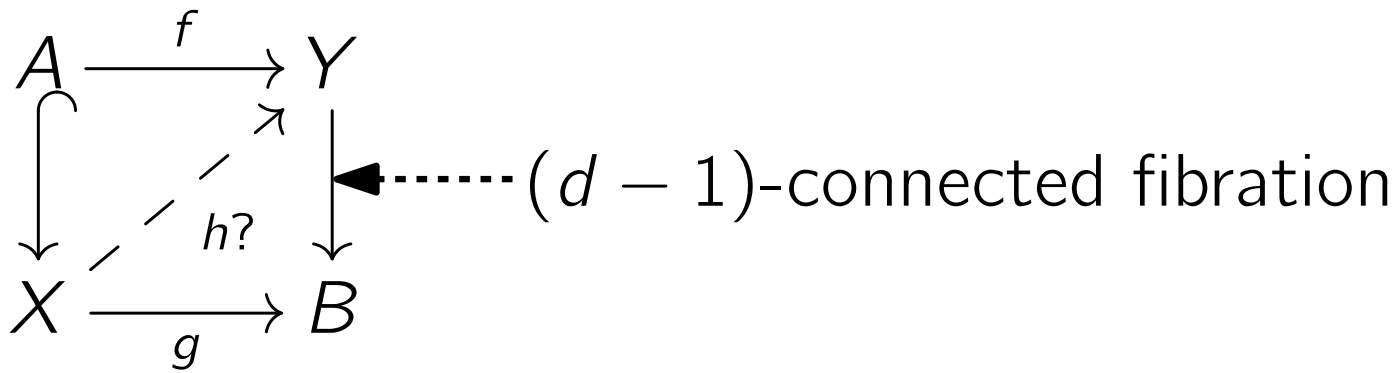

G-equivariant

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  - are there other ways? Can one generalize computation of Steenrod and Adem operations? Apply the deep knowledge of the stable homotopy groups of spheres?